NON-STRUCTURE IN λ^{++} USING INSTANCES OF WGCH

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§0 Introduction

Our aim is to prove the results of the form "build complicated/many models of cardinality λ^{++} by approximation of cardinality λ" assuming only $2^\lambda < 2^{\lambda^+} < 2^{\lambda^{++}}$, which are needed in developing classification of a.e.c., i.e. in this book and the related works (this covers [

Sh 87b], [Sh 88] redone in Chapter I and [Sh 576], [Sh 603] which are redone in Chapter VI + Chapter VII and [Sh 87b] which is redone by Chapter III, Chapter IV, [Sh 842], so we ignore, e.g. [Sh 576] now), fulfilling promises, uniformizing and correcting inaccuracies there and doing more. But en-route we spend time on the structure side.

As in [Sh 576, §3] we consider a version of construction framework, trying to give sufficient conditions for constructing many models of cardinality ∂^+ by approximations of cardinality $< \partial$ so λ^+ above correspond to ∂. Compared to [Sh 576, §3], the present version is hopefully more transparent.

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We start in §1,§2 (and also §3) by giving several sufficient conditions for non-structure, in a framework closer to the applications we have in mind than [Sh 576, §3]. The price is delaying the actual proofs and losing some generality. Later (mainly in §4, but also in §6 and §8) we do the applications, usually each is quoted (in some way) elsewhere. Of course, it is a delicate question how much should we repeat the background which exists when the quote was made.

The "many" is interpreted as $\geq \mu_{\rm unif}(\lambda^{++}, 2^{\lambda^+})$, see 9.4 why this is almost equal to $2^{\lambda^{++}}$. Unfortunately, there is here no one theorem covering all cases. But if a "lean" version suffice for us, which means that we assume the very weak set theoretic assumption "$\mathrm{WDmId}_{\lambda^+}$ (a normal ideal on λ^+) is not λ^{++}-saturated", (and, of course, we are content with getting $\geq \mu_{\rm unif}(\lambda^{++}, 2^{\lambda^+})$) <u>then</u> all the results can be deduced from weak coding, i.e. Theorem 2.3. In this case, some parts are redundant and the paper is neatly divided to two: structure part and non-structure part and we do now describe this.

First, in §1 we define a (so called) nice framework to deal with such theorems, and in the beginning of §2 state the theorem but we replace λ^+ by a regular uncountable cardinal ∂. This is done in a way closed to the applications we have in mind as deduced in §4. In Theorem 2.3 the model in ∂^+ is approximated triples by $(\bar M^\eta, \bar{\mathbf{J}}^\eta, \mathbf{f}^\eta)$ for $\eta \in {}^{\partial^+ >}2$ increasing with η where $\bar M^\eta$ is an increasing chain of length ∂ of models of cardinality $< \partial$ and for each $\eta \in {}^{\partial^+}2$ the sequence $\langle \cup\{M^{\eta \restriction \varepsilon}_\alpha : \alpha < \partial\} : \varepsilon < \partial^+\rangle$ is increasing; similarly in the other such theorems.

Theorem 2.3 is not proved in §2. It is proved in §9, §10, specifically in 10.10. Why? In the proof we apply relevant set theoretic results (see in the end of §0 and more in §9 on weak diamond and failure of strong uniformization), for this it is helpful to decide that the universe of each model (approximating the desired one) is $\subseteq \partial^+$ and to add commitments $\bar{\mathbb{F}}$ on the amalgamations used in the construction called amalgamation choice function.

So model theoretically they look artificial though the theorems are stronger.

Second, we deal with the applications in §4, actually in §4(A),(C),(D),(E), so we have in each case to choose $\mathfrak{u}$, the construction framework and prove the required properties. By our choice this goes naturally.

But we would like to eliminate the extra assumption "$\mathrm{WDmId}_{\lambda^+}$ is λ^{++}-saturated". So in §2 and §3 there are additional "coding" theorems. Some still need the "amalgamation choice function", others, as we have a stronger model theoretic assumption do not need such function so their proof is not delayed to §10.

Probably the most interesting case is proving the density of $K^{3,\rm uq}_{\mathfrak{s}}$, i.e. of uniqueness triples $(M,N,a) \in K^{3,\rm bs}_{\mathfrak{s}}$ for $\mathfrak{s}$ an (almost) good λ-frame, a somewhat weaker version of the (central) notion of Chapter III. Ignoring for a minute the "almost" this is an important step in (and is promised in) III§5. The proof is done in two stages. In the first stage we consider, in §6, a wider class $K^{3,\rm up}_{\mathfrak{s}} \subseteq K^{3,\rm bs}_{\mathfrak{s}}$ than $K^{3,\rm uq}_{\mathfrak{s}}$

and prove that its failure to be dense implies non-structure. This is done in §6, the proof is easier when $\lambda > \aleph_0$ or at least "$\mathscr{D}_{\lambda^+}$ is not λ^{++}-saturated"; (but this is unfortunate for an application to Chapter I). But for the proofs in §6 we need before this in §5 to prove some "structure positive theory" claims even if $\mathfrak{s}$ is a good λ-frame; we need more in the almost good case.

So naturally we assume (categoricity in λ and) density of $K^{3,\mathrm{up}}_{\mathfrak{s}}$ and prove (in §7) that $\mathrm{WNF}_{\mathfrak{s}}$, a weaker relative of $\mathrm{NF}_{\mathfrak{s}}$, is a weak $\mathfrak{s}$-non-forking relation on $\mathfrak{K}_{\mathfrak{s}}$ respecting $\mathfrak{s}$ and that $\mathfrak{s}$ is actually a good λ-frame (see 7.19(1)); both results are helpful.

The second stage (in §8) is done in two substages. In the first substage we deal with a delayed version of uniqueness, proving that its failure implies non-structure. In the second substage we assume delayed uniqueness but $K^{3,\mathrm{uq}}_{\mathfrak{s}}$ is not dense and we get another non-structure <u>but</u> relying on a positive consequence of density of $K^{3,\mathrm{up}}_{\mathfrak{s}}$ (that is, on a weak form of NF, see §7).

Why do we deal with almost good λ-frames? By III§3(E) from an a.e.c. $\mathfrak{K}$ categorical in λ, λ^+ which have $\in [1, \mu_{\mathrm{unif}}(\lambda^{++}, 2^{\lambda^+}))$ non-isomorphic models in λ^{++} we construct a good λ^+-frame $\mathfrak{s}$ with $\mathfrak{K}^{\mathfrak{s}} \subseteq \mathfrak{K}$. The non-structure theorem stated (and used) there is fully proven in §4. However, not only do we use the λ^{++}-saturation of the weak diamond ideal on λ^+, but it is a good λ^+-frame rather than a good λ-frame.

This does not hamper us in Chapter IV but still is regretable. Now we "correct" this but the price is getting an <u>almost</u> good λ-frame, noting that such $\mathfrak{s}$ is proved to exist in VI§8, the revised version of [Sh 576]. However, to arrive to those points in Chapter VI, [Sh 576] we have to prove the density of minimal types under the weaker assumptions, i.e. without the saturation of the ieal $\mathrm{WDmId}_{\lambda^+}$ together Chapter VII + VI§3,§4 gives a full proof. This requires again on developing some positive theory, so in §5 we do here some positive theory. Recall that [Sh 603], [Sh 576] are subsumed by them.

We can note that in building models $M \in K^{\mathfrak{s}}_{\lambda^{++}}$ for $\mathfrak{s}$ an almost good λ-frame, for convenience we use disjoint amalgamation. This may seem harmless but proving the density of the minimal triples this is not obvious; without assuming this we have to use $\langle M_\alpha, h_{\alpha,\beta} : \alpha < \beta < \partial \rangle$ instead of increasing $\langle M_\alpha : \alpha < \partial \rangle$; a notationally cumbersome choice. So we use a congruence relation $=_\tau$ but the models we construct are not what we need. We have to take their quotient by $=_\tau$, which has to, e.g. have the right cardinality. But we can take care that $|M|/\equiv_\tau$ has cardinality λ^{++} and $a \in M \Rightarrow |a/\equiv_\tau| = \lambda^{++}$. For the almost good λ-frame case this follows if we use not just models M which are λ^+-saturated above λ but if $M_0 \leq_{\mathfrak{K}[\mathfrak{s}]} M_1 \leq_{\mathfrak{K}[\mathfrak{s}]} M, M_0 \in K_{\mathfrak{s}}, M_1 \in K^{\mathfrak{s}}_{\leq \lambda}$ and $p \in \mathscr{S}^{\mathrm{bs}}_{\mathfrak{s}}(M_0)$ then for some $a \in M$, for every $M'_1 \leq_{\mathfrak{K}[\mathfrak{s}]} M_1$ of cardinality λ including M_0, the type $\mathbf{tp}_{\mathfrak{s}}(a, M'_1, M)$ is the non-forking extension of p, so not a real problem.

Reading Plans: The miser model - theorist Plan A:

If you like to see only the results quoted elsewhere (in this book), willing to assume an extra weak set theoretic assumption, this is the plan for you.

The results are all in §4, more exactly §4(A),(C),(D),(E). They all need only 2.3 relying on 2.2, but the rest of §2 and §3 are irrelevant as well as §5 - §8.

To understand what 2.3 say you have to read §1 (what is $\mathfrak{u}$; what are $\mathfrak{u}$-free rectangles; assuming $\mathfrak{K}$ is categorical in λ^+ you can ignore the "almost"). You may take 2.3 on belief, so you are done; otherwise you have to see §9 and 10.1 - 10.10.

The pure model - theorist Plan B:

Suitable if you like to know about the relatives of "good λ-frames". Generally see §5 - §8.

In particular on "almost good λ-frames" see §5; but better first read 1.1 - 1.14, which deal with a related framework called "nice construction framework" and in §6 learn of the class $K^{3,\mathrm{up}}_{\mathfrak{s}} \subseteq K^{3,\mathrm{bs}}_{\mathfrak{s}}$ with a weak version of uniqueness. By quoting we get non-structure if they fail density. Then in §7 learn on weak non-forking relations WNF on $\mathfrak{K}_\lambda$ which respects $\mathfrak{s}$, it is interesting when we assume $K^{3,\mathrm{up}}_{\mathfrak{s}}$ has density or reasonably weak existence assumption, because then we can prove that the definition given such existence, and this implies that $\mathfrak{s}$ is a good λ-frame (not just almost). In §8 we prove density of uinqueness triples ($K^{3,\mathrm{uq}}_{\mathfrak{s}}$) in $K^{3,\mathrm{bs}}_{\mathfrak{s}}$, so quote non-structure theorems.

The set theorist Plan C:

Read §1,§2,§3,§9,§10,§11 this presents construction in ∂^+ by approximation of cardinality $\le \lambda$.

0.1 Notation:

1) $\mathfrak{u}$, a construction framework, see §1, in particular Definition 1.2

2) Triples $(M, N, \mathbf{J}) \in \mathrm{FR}^\ell_{\mathfrak{u}}$, see Definition 1.2

2A) $\mathbf{J}$ and $\mathbf{I}$ are $\subseteq M \in K_{\mathfrak{u}}$

3) $\mathbf{d}$, and also $\mathbf{e}$, a $\mathfrak{u}$-free rectangle (or triangle), see Definitions 1.4, 1.6

4) $K^{\mathrm{qt}}_{\mathfrak{u}}$, the set of $(\bar{M}, \bar{\mathbf{J}}, \mathbf{f})$, see Definition 1.15, where, in particular:
   4A) $\mathbf{f}$ a function from ∂ to ∂
   4B) $\bar{\mathbf{J}} = \langle \mathbf{J}_\alpha : \alpha < \partial \rangle, \mathbf{J}_\alpha \subseteq M_{\alpha+1} \backslash M_\alpha$
   4C) $\bar{M} = \langle M_\alpha : \alpha < \partial \rangle$ where $M_\alpha \in \mathfrak{K}_{<\partial}$ is $\le_{\mathfrak{K}}$-increasing continuous

5) Orders (or relations) on $K^{\mathrm{qt}}_{\mathfrak{u}} : \le^{\mathrm{at}}_{\mathfrak{u}}, \le^{\mathrm{qt}}_{\mathfrak{u}}, \le^{\mathrm{qs}}_{\mathfrak{u}}$

6) $\mathbf{c}$, a colouring (for use in weak diamond)

7) $\mathbb{F}$ (usually $\bar{\mathbb{F}}$), for amalgamation choice functions, see Definition 10.3

8) $\mathfrak{g}$, a function from $K^{\rm qt}_{\mathfrak{u}}$ to itself, etc., see Definition 1.22
(for defining "almost every ...")
9) Cardinals $\lambda, \mu, \chi, \kappa, \theta, \partial$, but here $\partial = \text{ cf}(\partial) > \aleph_0$, see 1.8(2) in 1.8(1B), $\mathscr{D}_\partial$ is the club filter on ∂

10) $\mathbf{F}$ in the definition of limit model, see Definition Chapter I, marginal.

0.2 Definition. 1) For K a set or a class of models let $\dot I(K) = \{M/\cong : M \in K\}$, so it is a cardinality or ∞.
2) For a class K of models let $\dot I(\lambda, K) = \dot I(K_\lambda)$ where $K_\lambda = \{M \in K : \|M\| = \lambda\}$.
3) For $\mathfrak{K} = (K, \le_{\mathfrak{K}})$ let $\dot I(\mathfrak{K}) = I(K)$ and $\dot I(\lambda, \mathfrak{K}) = \dot I(\lambda, K)$.

Remark. We shall use in particular $\dot I(\mathfrak{K}^{\mathfrak{u},\mathfrak{h}}_{\partial^+})$, see Definition 1.23.

We now define some set theoretic notions (we use mainly the ideal WDmTId_∂ and the cardinals $\mu_{\rm wd}(\partial), \mu_{\rm unif}(\partial^+, 2^\partial)$).

0.3 Definition. Fix ∂ regular and uncountable.
1) For ∂ regular uncountable, $S \subseteq \partial$ and $\bar\chi = \langle \chi_\alpha : \alpha < \partial \rangle$ but only $\bar\chi \restriction S$ matters so we can use any $\bar\chi = \langle \chi_\alpha : \alpha \in S' \rangle$ where $S \subseteq S'$ let

$$\begin{aligned}\mathrm{WDmTId}(\partial, S, \bar\chi) = \Big\{ A : & A \subseteq \prod_{\alpha \in S} \chi_\alpha, \text{ and for some function } (= \text{colouring}) \\ & \mathbf{c} \text{ with domain } \bigcup_{\alpha < \partial} {}^\alpha(2^{<\partial}) \text{ mapping } {}^\alpha(2^{<\partial}) \text{ into } \chi_\alpha, \\ & \text{for every } \eta \in A, \text{ for some } f \in {}^\partial(2^{<\partial}) \text{ the set} \\ & \{\delta \in S : \eta(\delta) = \mathbf{c}(f \restriction \delta)\} \text{ is not stationary (in } \partial) \Big\}.\end{aligned}$$

(Note: WDmTId stands for weak diamond target ideal; of course, if we increase the χ_α we get a bigger ideal); the main case is when $\alpha \in S \Rightarrow \chi_\alpha = 2$ this is the weak diamond, see below.
1A) Here we can replace $2^{<\partial}$ by any set of this cardinality, and so we can replace $f \in {}^\partial(2^{<\partial})$ by $f_1, \dots, f_n \in {}^\partial(2^{<\partial})$ and $f \restriction \delta$ by $\langle f_1 \restriction \delta, \dots, f_n \restriction \delta \rangle$ and $\mathbf{c}(f \restriction \delta)$ by $\mathbf{c}'(f_1 \restriction \delta, \dots, f_n \restriction \delta)$ so with $\mathbf{c}'$ being an n-place function; justified in [Sh:f, AP,§1].
2)[1]

[1] in [Sh:b, AP,§1], [Sh:f, AP,§1] we express $\mathrm{cov}_{\rm wdmt}(\partial, S) > \mu^*$ by allowing $f(0) \in \mu^* < \mu$

$$\mathrm{cov}_{\mathrm{wdmt}}(\partial, S, \bar{\chi}) = \mathrm{Min}\Big\{|\mathscr{P}| : \mathscr{P} \subseteq \mathrm{WDmTId}(\partial, S, \bar{\chi}) \text{ and } \prod_{\alpha \in S} \chi_\alpha \subseteq \bigcup_{A \in \mathscr{P}} A\Big\}$$

3) $$\mathrm{WDmTId}_{<\mu}(\partial, S, \bar{\chi}) = \Big\{A \subseteq \prod_{\alpha \in S} \chi_\alpha : \text{for some } i^* < \mu \text{ and } A_i \in \mathrm{WDmTId}(\partial, S, \bar{\chi}) \text{ for } i < i^* \text{ we have } A \subseteq \bigcup_{i<i^*} A_i\Big\}$$

4) (a) $$\mathrm{WDmTId}(\partial) = \mathrm{WDmTId}(\partial, \partial, 2)$$

4) (b) $$\mathrm{WDmId}_{<\mu}(\partial, \bar{\chi}) = \{S \subseteq \partial : \mathrm{cov}_{\mathrm{wdmt}}(\partial, S, \bar{\chi}) < \mu\}.$$

5) Instead of "$< \mu^+$" we may write $\leq \mu$ or just μ; if we omit μ we mean $(2^{<\partial})$. If $\bar{\chi}$ is constantly 2 we may omit it, see below, if $\chi_\alpha = 2^{|\alpha|}$ we may write pow instead of $\bar{\chi}$; all this in the parts above and below.

6) Let $\mu_{\mathrm{wd}}(\partial, \bar{\chi}) = \mathrm{cov}_{\mathrm{wdmt}}(\partial, \partial, \bar{\chi})$.

7) We say that the weak diamond holds on λ <u>if</u> $\partial \notin \mathrm{WDmId}(\partial)$.

Remark. This is used in E46-3c.22 VI. E46-3c.22, VI. E46-6f.13. Note that by 0.5(1A) that $\mu_{\mathrm{wd}}(\lambda^+)$ is large (but $\leq 2^{\lambda^+}$, of course).

A relative is

0.4 Definition. Fix ∂ regular and uncountable.
1) For ∂ regular uncountable, $S \subseteq \partial$ and $\bar{\chi} = \langle \chi_\alpha : \alpha < \partial \rangle$ let

$$\mathrm{UnfTId}(\partial, S, \bar{\chi}) = \Big\{A : A \subseteq \prod_{\alpha \in S} \chi_\alpha, \text{ and for some function } (= \text{colouring}) \ \mathbf{c} \text{ with domain } \bigcup_{\alpha<\partial} {}^{\alpha}(2^{<\partial}) \text{ mapping } {}^{\alpha}(2^{<\partial}) \text{ into } \chi_\alpha, \text{ for every } \eta \in A, \text{ for some } f \in {}^{\partial}(2^{<\partial}) \text{ the set } \{\delta \in S : \eta(\delta) \neq \mathbf{c}(f \restriction \delta)\} \text{ is not stationary (in } \partial)\Big\}.$$

(Note: UnfTId stands for uniformization target ideal; of course, if we increase the χ_α we get a smaller ideal); when $\alpha \in S \Rightarrow \chi_\alpha = 2$ this is the weak diamond, i.e. as in 0.3(1), similarly below.
1A) Also here we can replace $2^{<\partial}$ by any set of this cardinality, and so we can replace $f \in {}^\partial(2^{<\partial})$ by $f_1, \ldots, f_n \in {}^\partial(2^{<\partial})$ and $f \restriction \delta$ by $\langle f_1 \restriction \delta, \ldots, f_n \restriction \delta\rangle$ and $\mathbf{c}(f \restriction \delta)$ by $\mathbf{c}'(f_1 \restriction \delta, \ldots, f_n \restriction \delta)$ so with $\mathbf{c}'$ being an n-place function, justified in [Sh:f, AP,§1].

2) $$\mathrm{cov}_{\mathrm{unf}}(\partial, S, \bar{\chi}) = \ \mathrm{Min}\Big\{|\mathscr{P}| : \mathscr{P} \subseteq \ \mathrm{UnfTId}(\partial, S, \bar{\chi}) \text{ and } \prod_{\alpha \in S} \chi_\alpha \subseteq \bigcup_{A \in \mathscr{P}} A\Big\}$$

3) $$\mathrm{UnfTId}_{<\mu}(\partial, S, \bar{\chi}) = \Big\{A \subseteq \prod_{\alpha \in S} \chi_\alpha : \text{for some } i^* < \mu \text{ and } A_i \in \ \mathrm{UnfTId}(\partial, S, \bar{\chi}) \text{ for } i < i^* \text{ we have } A \subseteq \bigcup_{i < i^*} A_i\Big\}$$

4) $$\mathrm{UnfId}_{<\mu}(\partial, \bar{\chi}) = \Big\{S \subseteq \partial : \mathrm{cov}_{\mathrm{unf}}(\partial, S, \bar{\chi}) < \mu\Big\}.$$

5) Instead of "$< \mu^+$" we may write $\leq \mu$ or just μ, if we omit μ we mean $(2^{<\partial})$. If $\bar{\chi}$ is constantly 2 we may omit it, if $\chi_\alpha = 2^{|\alpha|}$ we may write pow instead of $\bar{\chi}$; all this in the parts above and below.
6) $\mu_{\mathrm{unif}}(\partial, \bar{\chi})$ where $\bar{\chi} = \langle \chi_\alpha : \alpha < \partial\rangle$ is $\mathrm{Min}\{|\mathscr{P}| : \mathscr{P}$ is a family of subsets of $\prod\limits_{\alpha<\partial} \chi_\alpha$ with union $\prod\limits_{\alpha<\partial} \chi_\alpha$ and for each $A \in \mathscr{P}$ there is a function $\mathbf{c}$ with domain $\bigcup\limits_{\alpha<\partial} \prod\limits_{\beta<\alpha} \chi_\beta$ such that $f \in A \Rightarrow \{\delta \in S : \mathbf{c}(f \restriction \delta) = f(\delta)\}$ is not stationary$\}$.
7) $\mu_{\mathrm{unif}}(\partial, \chi) = \mu_{\mathrm{unif}}(\partial, \bar{\chi})$ where $\bar{\chi} = \langle \chi : \alpha < \partial\rangle$ and $\mu_{\mathrm{unif}}(\partial, < \chi)$ means $\sup\{\mu_{\mathrm{unif}}(\partial, \chi_1) : \chi_1 < \chi\}$; similarly in the other definitions above. If $\chi = 2$ we may omit it.

By Devlin Shelah [DvSh 65], [Sh:b, XIV,1.5,1.10](2);1.18(2),1.9(2) (presented better in [Sh:f, AP,§1] we have:

0.5 Theorem.
1) If $\partial = \aleph_1, 2^{\aleph_0} < 2^{\aleph_1}, \mu \le (2^{\aleph_0})^+$ *then* $\partial \notin \mathrm{WDmId}_{<\mu}(\partial)$.
2) If $2^\theta = 2^{<\partial} < 2^\partial, \mu = (2^\theta)^+$*, or just: for some* $\theta, 2^\theta = 2^{<\partial} < 2^\partial, \mu \le 2^\partial$*, and* $\chi^\theta < \mu$ *for* $\chi < \mu$*, then* $\partial \notin \mathrm{WDmId}_\mu(\partial)$ *equivalently* ${}^\partial 2 \notin \mathrm{WDmTId}_\mu(\partial)$*. So* $(\mu_{\mathrm{wd}}(\partial))^\theta = 2^\partial$.
3) Assume $2^\theta = 2^{<\partial} < 2^\partial, \mu \le 2^\partial$ *and*

- (a) $\mu \le \partial^+$ *or* $\mathrm{cf}([\mu_1]^{\le \partial}, \subseteq) < \mu$ *for* $\mu_1 < \mu$ *and*
- (b) $\mathrm{cf}(\mu) > \partial \, \mathrm{or} \, \mu \le (2^{<\partial})^+$.

Then $\mathrm{WDmId}_{<\mu}(\partial, \bar{\chi})$ *is a normal ideal on* ∂ *and* $\mathrm{WDmTId}_{<\mu}(\partial, \chi)$ *is a* $\mathrm{cf}(\mu)$*-complete ideal on* ${}^\partial 2$*. [If this ideal is not trivial, then* $\partial = \mathrm{cf}(\partial) > \aleph_0, 2^{<\partial} < 2^\partial$*.]*
4) $\mathrm{WDmTId}_{<\mu}(\partial, S, \bar{\chi})$ *is* $\mathrm{cf}(\mu)$*-complete ideal on* $\prod\limits_{\alpha \in S} \chi_\alpha$.

0.6 Remark. 0) Compare to §9,§10, mainly 9.6.
1) So if $\mathrm{cf}(2^\partial) < \mu$ (which holds if 2^∂ is singular and $\mu = 2^\partial$) then 0.5(3) implies that there is $A \subseteq {}^\partial 2, |A| < 2^\partial, A \notin \mathrm{WDmTId}(\partial)$.
2) Some related definitions appear in [Sh:E45, §1], mainly $\mathrm{DfWD}_{<\mu}(\partial)$, but presently we ignore them.
3) We did not look again at the case $(\forall \sigma < \lambda)(2^\sigma < 2^{<\partial} < 2^\partial)$.
4) Recall that for an a.e.c. $\mathfrak{K}$:

- (a) if $K_\lambda \ne \emptyset$ but $\mathfrak{K}$ has no $\le_{\mathfrak{K}}$-maximal model in K_λ then $K_{\lambda^+} \ne \emptyset$
- (b) if $\mathfrak{K}$ is categorical in λ and $\mathrm{LS}(\mathfrak{K}) \le \lambda$ then $K_{\lambda^+} \ne \emptyset$ iff K has no $\le_{\mathfrak{K}}$-maximal model in K_λ.

5) About $\mu_{\mathrm{wd}}(\partial)$ see VI. E46-1a.16, VI. E46-2b.13, VI. E46-6f.4.

0.7 Definition. 1) We say that a normal ideal $\mathbb{I}$ on a regular uncountable cardinal λ is μ-saturated when we cannot find a sequence $\bar{A} = \langle A_i : i < \mu \rangle$ such that $A_i \subseteq \lambda, A_i \notin \mathbb{I}$ for $i < \mu$ and $A_i \cap A_j \in \mathbb{I}$ for $i \ne j < \mu$; if $\mu \le \lambda^+$ without loss of generality $A_i \cap A_j \in [\lambda]^{<\lambda}$.
2) Similarly for a normal filter on a regular uncountable cardinal λ.

§1 Nice contruction framework

We define here when $\mathfrak{u}$ is a nice construction framework. Now $\mathfrak{u}$ consists of an a.e.c. $\mathfrak{K}$ with $\mathrm{LS}(\mathfrak{K}) < \partial_{\mathfrak{u}} = \mathrm{cf}(\partial_{\mathfrak{u}})$, and enables us to build a model in K_{∂^+} by approximations of cardinality $< \partial := \partial_{\mathfrak{u}}$.

Now for notational reasons we prefer to use increasing sequences of models rather than directed systems, i.e., sequences like $\langle M_\alpha, f_{\beta,\alpha} : \alpha \le \beta < \alpha^* \rangle$ with $f_{\beta,\alpha} : M_\alpha \to M_\beta$ satisfying $f_{\gamma,\beta} \circ f_{\beta,\alpha} = f_{\alpha,\gamma}$ for $\alpha \le \beta \le \gamma < \alpha^*$. For this it is very desirable to have disjoint amalgamation; however, in one of the major applications (the density of minimal types, see here in §4(A),(B) or in [Sh 576, §3] used in VI§3,§4) we do not have this. In [Sh 576, §3] the solution was to allow non-standard interpretation of the equality (see Definition 1.10 here). Here we choose another formulation: we have $\tau \subseteq \tau(\mathfrak{u})$ such that we are interested in the non-isomorphism of the τ-reducts $M^{[\tau]}$ of the M's constructed, see Definition 1.8. Of course, this is only a notational problem.

The main results on such $\mathfrak{u}$ appear later; a major theorem is 2.3, deducing non-structure results assuming the weak coding property. This and similar theorems, assuming other variant of the coding property, are dealt with in §2,§3. They all have (actually lead to) the form "if most triples $(\bar{M}, \bar{\mathbf{J}}, \mathbf{f}) \in K^{\mathrm{qt}}_{\mathfrak{u}}$ has, in some sense 2 (or many, say $2^{<\partial}$) extensions which are (pairwise) incompatible in suitable sense, <u>then</u> we build a suitable tree $\langle (\bar{M}_\eta, \bar{\mathbf{J}}_\eta, f_\eta) : \eta \in {}^{\partial^+ >}2 \rangle$ and letting $M_\eta = \cup\{M^\eta_\alpha : \alpha < \lambda\}$ for $\eta \in {}^{\partial^+ >}2$ and $M_\nu := \cup\{M_{\nu \restriction \alpha} : \alpha < \partial^+\}$ for $\nu \in {}^{\partial^+}2$ we have: among $\langle M_\nu : \nu \in {}^{\partial^+}2 \rangle$ many are non-isomorphic (and in K_{∂^+}). Really, usually the indexes are $\eta \in {}^{\partial^+ >}(2^\partial)$ and the conditions speak on amalgamation in $\mathfrak{K}_{\mathfrak{u}}$, i.e. on models of cardinality $< \partial$ but using FR_1, FR_2, see below.

As said earlier, in the framework defined below we (relatively) prefer transparency and simplicity on generality, e.g. we can weaken "$\mathfrak{K}_{\mathfrak{u}}$ is an a.e.c." and/or make FR^+_ℓ is axiomatic and/or use more than atomic successors (see 10.14 + 10.16).

In 1.1 - 1.6 we introduce our frameworks $\mathfrak{u}$ and $\mathfrak{u}$-free rectangles/triangles; in 1.7 - the dual of $\mathfrak{u}$, and in 1.8 - 1.12 we justify the disjoint amalgamation through "τ is a $\mathfrak{u}$-sub-vocabulary", so a reader not bothered by this point can ignore it, then in 1.13 we consider another property of $\mathfrak{u}$, monotonicity and in 1.14 deal with variants of $\mathfrak{u}$.

In 1.15 - 1.26 we introduce a class $K^{\mathrm{qt}}_{\mathfrak{u}}$ of triples $(\bar{M}, \bar{\mathbf{J}}, \mathbf{f})$ serving as approximations of size ∂, some relations and orders on it and variants, and define what it means "for almost every such triple" (if $K_{\mathfrak{u}}$ is categorical in ∂ this is usually easy and in many of our applications for most $(\bar{M}, \bar{\mathbf{J}}, \mathbf{f})$ the model $\cup\{M_\alpha : \alpha < \partial\}$ is saturated (of cardinality ∂).

<u>1.1 Convention</u>: If not said otherwise, $\mathfrak{u}$ is as in Definition 1.2.

1.2 Definition. We say that $\mathfrak{u}$ is a nice construction framework <u>when</u> (the demands are for $\ell = 1, 2$ and later (D) means $(\mathrm{D})_1$ and $(\mathrm{D})_2$ and (E) means $(\mathrm{E})_1$ and $(\mathrm{E})_2$):

- (A) $\mathfrak{u}$ consists of $\partial, \mathfrak{K} = (K, \leq_{\mathfrak{K}}), \mathrm{FR}_1, \mathrm{FR}_2, \leq_1, \leq_2$ (also denoted by $\partial_{\mathfrak{u}}, \mathfrak{K}^{\mathfrak{u}} = \mathfrak{K}^{\mathrm{up}}_{\mathfrak{u}} = (K^{\mathrm{up}}_{\mathfrak{u}}, \leq_{\mathfrak{u}}), \mathrm{FR}^{\mathfrak{u}}_1, \mathrm{FR}^{\mathfrak{u}}_2, \leq^1_{\mathfrak{u}}, \leq^2_{\mathfrak{u}}$) and let $\tau_{\mathfrak{u}} = \tau_{\mathfrak{K}}$. The indexes 1 and 2 can be replaced by ver (vertical[2], direction of ∂) and hor (horizontal, direction of ∂^+) respectively
- (B) ∂ is regular uncountable
- (C) $\mathfrak{K} = \mathfrak{K}^{\mathrm{up}}_{\mathfrak{u}} = (K, \leq_{\mathfrak{K}})$ is an a.e.c., $K \neq \emptyset$ of course with $\mathrm{LS}(\mathfrak{K}) < \partial$ (or just $(\forall M \in K)(\forall A \in [M]^{<\partial})(\exists N)(A \subseteq N \leq_{\mathfrak{K}} M \wedge \|N\| < \partial)$). Let $\mathfrak{K}_{\mathfrak{u}} = \mathfrak{K}_{<\partial} = (\mathfrak{K}_{<\partial}, \leq_{\mathfrak{K}} \restriction K_{<\partial})$ where $K_{<\partial} = K_{\mathfrak{u}} \restriction \{M : M \in K^{\mathrm{up}}_{\mathfrak{u}}$ has cardinality $< \partial\}$ and $\mathfrak{K}[\mathfrak{u}] = K^{\mathrm{up}}_{\mathfrak{u}}$.
  (To prepare for weaker versions we can start with $\mathfrak{K}_{\mathfrak{u}}$, a $(< \partial)$-a.e.c.[3]; this means K is a class of models of cardinality $< \partial$, and in AxIII, the existence of union we add the assumption that the length of the union is $< \partial$ (here equivalently the union has cardinality $< \partial$) and we replace "$\mathrm{LS}(\mathfrak{K}_{<\partial})$ exists" by $K_{<\partial} \neq \emptyset$ and let $\mathfrak{K}^{\mathrm{up}}_{\mathfrak{u}}$ be its lifting up, as in III. 600-0.31 and we assume $K \neq \emptyset$ so $\mathfrak{K} = \mathfrak{K}^{\mathrm{up}}_{\mathfrak{u}}$ and we write $\tau_{\mathfrak{u}} = \tau_{\mathfrak{K}}$ and $\leq_{\mathfrak{K}}$ for $\leq_{\mathfrak{K}^{\mathrm{up}}}$ and $\leq_{\mathfrak{u}}$ for $\leq_{\mathfrak{K}_{<\partial}}$)
- $(D)_\ell$ (a) FR_ℓ is a class of triples of the form $(M, N, \mathbf{J})$, closed under isomorphisms, let $\mathrm{FR}^+_\ell = \mathrm{FR}^{\mathfrak{u},+}_\ell$ be the family of $(M, N, \mathbf{J}) \in \mathrm{FR}_\ell$ such that $\mathbf{J} \neq \emptyset$;
  - (b) if $(M, N, \mathbf{J}) \in \mathrm{FR}_\ell$ <u>then</u> $M \leq_{\mathfrak{u}} N$ hence both are from $K_{\mathfrak{u}}$ so of cardinality $< \partial$
  - (c) if $(M, N, \mathbf{J}) \in \mathrm{FR}_\ell$ <u>then</u>[4] $\mathbf{J}$ is a set of elements of $N \backslash M$
  - (d) if $M \in K_{\mathfrak{u}}$ <u>then</u>[5] for some $N, \mathbf{J}$ we have $(M, N, \mathbf{J}) \in \mathrm{FR}^+_\ell$
  - (e) if $M \leq_{\mathfrak{u}} N \in K_{\mathfrak{u}}$ <u>then</u>[6] $(M, N, \emptyset) \in \mathrm{FR}^{\mathfrak{u}}_\ell$

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[2]Hard but immaterial choice. We construct a model of cardinality ∂^+ by a sequence of length ∂^+ approximations, each of the form $\langle M_\alpha, \mathbf{J}_\alpha : \alpha < \partial\rangle, M_\alpha \in K_{<\partial}$ is $\leq_{\mathfrak{K}_{<\partial}}$-increasing and $(M_\alpha, M_{\alpha+1}, \mathbf{J}_\alpha) \in \mathrm{FR}_2$. If $\langle M'_\alpha, \mathbf{J}'_\alpha : \alpha < \partial\rangle$ is an immediate successor in the ∂^+-direction of $\langle M_\alpha, \mathbf{J}_\alpha : \alpha < \partial\rangle$ then for most $\alpha, M_\alpha \leq_{\mathfrak{u}} M'_\alpha$ and $(M_\alpha, M'_\alpha, \mathbf{I}_\alpha) \in \mathrm{FR}_1$ for suitable $\mathbf{I}_\alpha$, increasing with α and $(M'_\alpha, M'_{\alpha+1}, \mathbf{J}'_\alpha) \in \mathrm{FR}_2$ is $\leq^2_{\mathfrak{u}}$-above $(M_\alpha, M_{\alpha+1}, \mathbf{J}_\alpha)$. Now the natural order on FR_2 leads in the horizontal direction.

[3]less is used, but natural for our applications, see §9

[4]If we use the a.e.c. $\mathfrak{K}'$ defined in 1.10 we, in fact, weaken this demand to "$\mathbf{J} \subseteq N$". This is done, e.g. in the proof of 4.1 that is in Definition 4.5.

[5]We can weaken this and in some natural example we have less, but we circumvent this, via 1.10, see 4.1(c); this applies to (E)(b)(β), too; see Example 2.8

[6]not a great loss if we demand $M = N$; but then we have to strengthen the amalgamation demand (clause (F)); this is really needed only for $\ell = 2$

$(E)_\ell$ (a) $\le_\ell = \le^\ell_{\mathfrak{u}}$ is a partial order on FR_ℓ, closed under isomorphisms

$(b)(\alpha)$ if $(M_1, N_1, \mathbf{J}_1) \le_\ell (M_2, N_2, \mathbf{J}_2)$ then $M_1 \le_{\mathfrak{u}} M_2, N_1 \le_{\mathfrak{u}} N_2$ and $\mathbf{J}_1 \subseteq \mathbf{J}_2$

(β) moreover $N_1 \cap M_2 = M_1$ (disjointness)

(c) if $\langle (M_i, N_i, \mathbf{J}_i) : i < \delta \rangle$ is $\le_\ell$-increasing continuous (i.e. in limit we take unions) and $\delta < \partial$ <u>then</u> the union $\bigcup\limits_{i<\delta}(M_i, N_i, \mathbf{J}_i) := (\bigcup\limits_{j<\delta} M_i, \bigcup\limits_{i<\delta} N_i, \bigcup\limits_{i<\delta} \mathbf{J}_i)$ belongs to FR_ℓ and $j < \delta \Rightarrow (M_j, N_j, \mathbf{J}_j) \le_\ell (\bigcup\limits_{i<\delta} M_i, \bigcup\limits_{i<\delta} N_i, \bigcup\limits_{i<\delta} \mathbf{J}_i)$

(d) if $M_1 \le_{\mathfrak{u}} M_2 \le_{\mathfrak{u}} N_2$ and $M_1 \le_{\mathfrak{u}} N_1 \le_{\mathfrak{u}} N_2$ <u>then</u> $(M_1, N_1, \emptyset) \le_\ell (M_2, N_2, \emptyset)$

(F) (amalgamation) if $(M_0, M_1, \mathbf{I}_1) \in \mathrm{FR}_1, (M_0, M_2, \mathbf{J}_1) \in \mathrm{FR}_2$ and $M_1 \cap M_2 = M_0$ <u>then</u> we can find[7] $M_3, \mathbf{I}_2, \mathbf{J}_2$ such that $(M_0, M_1, \mathbf{I}_1) \le_1 (M_2, M_3, \mathbf{I}_2)$ and $(M_0, M_2, \mathbf{J}_1) \le_2 (M_1, M_3, \mathbf{J}_2)$ hence $M_\ell \le_{\mathfrak{u}} M_3$ for $\ell = 0, 1, 2$.

1.3 Claim. *1) $\mathfrak{K}_{\mathfrak{u}}$ has disjoint amalgamation.*
2) If $\ell = 1, 2$ and $(M_0, M_1, \mathbf{I}_1) \in \mathrm{FR}_\ell$ and $M_0 \le_{\mathfrak{u}} M_2$ and $M_1 \cap M_2 = M_0$ <u>then</u> we can find a pair $(M_3, \mathbf{I}^*_2)$ such that: $(M_0, M_1, \mathbf{I}) \le^\ell_{\mathfrak{u}} (M_2, M_3, \mathbf{I}^*_2) \in \mathrm{FR}^{\mathfrak{u}}_\ell$.

Proof. 1) Let $M_0 \le_{\mathfrak{K}_{\mathfrak{u}}} M_\ell$ for $\ell = 1, 2$ and for simplicity $M_1 \cap M_2 = M_0$. Let $\mathbf{I}_1 = \emptyset$, so by condition $(\mathrm{D})_1$(e) of Definition 1.2 we have $(M_0, M_1, \mathbf{I}_1) \in \mathrm{FR}_1$. Now apply part (2), (for $\ell = 1$).
2) By symmetry without loss of generality $\ell = 1$. Let $\mathbf{J}_1 := \emptyset$, so by Condition $(\mathrm{D})_2$(e) of Definition 1.2 we have $(M_0, M_2, \mathbf{J}_1) \in \mathrm{FR}^{\mathfrak{u}}_2$. So $M_0, M_1, \mathbf{I}_1, M_2, \mathbf{J}_1$ satisfies the assumptions of condition (F) of Definition 1.2 hence there are $M_3, \mathbf{I}_2, \mathbf{J}_2$ as guaranteed there so in particular $(M_0, M_1, \mathbf{I}_1) \le^2_{\mathfrak{u}} (M_2, M_3, \mathbf{I}_2)$ so the pair $(M_3, \mathbf{I}_2)$ is as required. $\square_{1.3}$

1.4 Definition. 1) We say that $\mathbf{d}$ is a $\mathfrak{u}$-free (α, β)-rectangle or is $\mathfrak{u}$-non-forking (α, β)-rectangle (we may omit $\mathfrak{u}$ when clear from the context) <u>when</u>:

(a) $\mathbf{d} = (\langle M_{i,j} : i \le \alpha, j \le \beta \rangle, \langle \mathbf{J}_{i,j} : i < \alpha, j \le \beta \rangle, \langle \mathbf{I}_{i,j} : i \le \alpha, j < \beta \rangle)$, (we may add superscript $\mathbf{d}$, the "$i < \alpha$", "$j < \beta$" are not misprints)

(b) $\langle (M_{i,j}, M_{i,j+1}, \mathbf{I}_{i,j}) : i \le \alpha \rangle$ is $\le_1$-increasing continuous for each $j < \beta$

[7] we can ask for $M'_3 \le_{\mathfrak{K}} M''_3$ and demand $(M_0, M_1, \mathbf{J}_1) \le_2 (M_2, M'_3, \mathbf{J}'_1), (M_0, M_2, \mathbf{I}_1) \le (M_1, M''_3, \mathbf{I}'_2)$, no real harm here but also no clear gain

(c) $\langle (M_{i,j}, M_{i+1,j}, \mathbf{J}_{i,j}) : j \leq \beta \rangle$ is $\leq_2$-increasing continuous for each $i < \alpha$

(d) $M_{\alpha,\beta} = \cup\{M_{i,\beta} : i < \alpha\}$ if α, β are limit ordinals.

2) For $\mathbf{d}^1$ a $\mathfrak{u}$-free (α, β)-rectangle and $\alpha_1 \leq \alpha, \beta_1 \leq \beta$ let $\mathbf{d}^2 = \mathbf{d}^1 \restriction (\alpha_1, \beta_1)$ means:

(a) $\mathbf{d}^2$ is a $\mathfrak{u}$-free (α_1, β_1)-rectangle, see 1.5 below

(b) the natural equalities: $M^{\mathbf{d}^1}_{i,j} = M^{\mathbf{d}^2}_{i,j}, \mathbf{J}^{\mathbf{d}^1}_{i,j} = \mathbf{J}^{\mathbf{d}^2}_{i,j}, \mathbf{I}^{\mathbf{d}_1}_{i,j} = \mathbf{I}^{\mathbf{d}_2}_{i,j}$ when both sides are well defined.

3) $\mathbf{d}^2 = \mathbf{d}^1 \restriction ([\alpha_1, \alpha_2], [\beta_1, \beta_2])$ when $\alpha_1 \leq \alpha_2 \leq \alpha, \beta_1 \leq \beta_2 \leq \beta$ is defined similarly.
4) For $\mathbf{d}$ as above we may also write $\alpha_{\mathbf{d}}, \alpha(\mathbf{d})$ for α and $\beta_{\mathbf{d}}, \beta(\mathbf{d})$ for β and if $\mathbf{I}^{\mathbf{d}}_{i,j}$ is a singleton we may write $\mathbf{I}^{\mathbf{d}}_{i,j} = \{a^{\mathbf{d}}_{i,j}\}$ and may just write $(M^{\mathbf{d}}_{i,j}, M^{\mathbf{d}}_{i,j+1}, a^{\mathbf{d}}_{i,j})$ and if $\mathbf{J}^{\mathbf{d}}_{i,j}$ is a singleton we may write $\mathbf{J}^{\mathbf{d}}_{i,j} = \{b^{\mathbf{d}}_{i,j}\}$ and may write $(M^{\mathbf{d}}_{i,j}, M^{\mathbf{d}}_{i+1,j}, b^{\mathbf{d}}_{i,j})$. Similarly in Definition 1.6 below.
5) We may allow $\alpha = \partial$ and or $\beta \leq \partial$, but we shall say this.

1.5 Observation. 1) The restriction in Definition 1.4(2) always gives a $\mathfrak{u}$-free (α_1, β_1)-rectangle.
2) The restriction in Definition 1.4(3) always gives a $\mathfrak{u}$-free $(\alpha_2 - \alpha_1, \beta_2 - \beta_1)$-rectangle.
3) If $\mathbf{d}$ is a $\mathfrak{u}$-free (α, β)-rectangle, <u>then</u>

(e) $\langle M^{\mathbf{d}}_{i,j} : i \leq \alpha \rangle$ is $\leq_{\mathfrak{u}}$-increasing continuous for $j \leq \beta$

(f) $\langle M^{\mathbf{d}}_{i,j} : j \leq \beta \rangle$ is $\leq_{\mathfrak{u}}$-increasing continuous for each $i \leq \alpha$.

4) In Definition 1.6 below, clause (d), when $j < \beta$ or $j = \beta \wedge (\beta$ successor) follows from (b). Similarly for the pair of clauses (e),(c).
5) Assume that $\alpha_1 \leq \alpha_2, \beta_1 \geq \beta_2$ and $\mathbf{d}_\ell$ is $\mathfrak{u}$-free $(\alpha_\ell, \beta_\ell)$-rectangle for $\ell = 1, 2$ and $\mathbf{d}_1 \restriction (\alpha_1, \beta_2) = \mathbf{d}_2 \restriction (\alpha_1, \beta_2)$ and $M^{\mathbf{d}_1}_{\alpha_1,\beta_1} \cap M^{\mathbf{d}_2}_{\alpha_2,\beta_2} = M^{\mathbf{d}_\ell}_{\alpha_1,\beta_2}$. <u>Then</u> we can find a $\mathfrak{u}$-free (α_2, β_1)-rectangle $\mathbf{d}$ such that $\mathbf{d} \restriction (\alpha_\ell, \beta_\ell) = \mathbf{d}_\ell$.

Proof. Immediate, e.g. in (5) we use clause (F) of Definition 1.2 for each $\alpha \in [\alpha_1, \alpha_2), \beta \in [\beta_2, \beta_1)$ in a suitable induciton. $\square_{1.5}$

1.6 Definition. We say that $\mathbf{d}$ is a $\mathfrak{u}$-free $(\bar{\alpha}, \beta)$-triangle or $\mathfrak{u}$-non-forking $(\bar{\alpha}, \beta)$-triangle <u>when</u> $\bar{\alpha} = \langle \alpha_i : i \leq \beta \rangle$ is a non-decreasing[8] sequence of ordinals and (letting $\alpha := \alpha_\beta$):

[8] not unreasonable to demand $\bar{\alpha}$ to be increasing continuous

(a) $\mathbf{d} = (\langle M^{\mathbf{d}}_{i,j} : i \le \alpha_j, j \le \beta\rangle, \langle \mathbf{J}_{i,j} : i < \alpha_j, j \le \beta\rangle, \langle \mathbf{I}_{i,j} : i \le \alpha_j, j < \beta\rangle)$

(b) $\langle (M^{\mathbf{d}}_{i,j}, M^{\mathbf{d}}_{i,j+1}, \mathbf{I}_{i,j}) : i \le \alpha_j\rangle$ is $\le_1$-increasing continuous for each $j < \beta$

(c) $\langle (M^{\mathbf{d}}_{i,j}, M^{\mathbf{d}}_{i+1,j}, \mathbf{J}_{i,j}) : j \le \beta$ and j is such that $i+1 \le \alpha_j\rangle$ is $\le_2$-increasing continuous for each $i < \alpha$

(d) for each $j \le \beta$ the sequence $\langle M_{i,j} : i \le \alpha_j\rangle$ is $\le_{\mathfrak{u}}$-increasing continuous

(e) for each $i_* \in [\alpha_{j_*}, \alpha), j_* \le \beta$ the sequence $\langle M_{i_*,j} : j \in [j_*, \beta]\rangle$ is $\le_{\mathfrak{K}}$-increasing continuous.

1.7 Definition/Claim. 1) For nice construction framework $\mathfrak{u}_1$ let $\mathfrak{u}_2 = \mathrm{dual}(\mathfrak{u}_1)$ be the unique nice construction framework $\mathfrak{u}_2$ such that: $\partial_{\mathfrak{u}_2} = \partial_{\mathfrak{u}_1}, \mathfrak{K}_{\mathfrak{u}_2} = \mathfrak{K}_{\mathfrak{u}_1}$ (hence $\mathfrak{K}^{\mathrm{up}}_{\mathfrak{u}_2} = K^{\mathrm{up}}_{\mathfrak{u}_1}$, etc) and $(\mathrm{FR}^{\mathfrak{u}_2}_\ell, \le^\ell_{\mathfrak{u}_2}) = (\mathrm{FR}^{\mathfrak{u}_1}_{3-\ell}, \le^{3-\ell}_{\mathfrak{u}_1})$ for $\ell = 1,2$.
2) We call $\mathfrak{u}_1$ self-dual when $\mathrm{dual}(\mathfrak{u}_1) = \mathfrak{u}_1$.
3) In part (1), if addition if $\mathbf{d}_1$ is $\mathfrak{u}_1$-free rectangle <u>then</u> there is a unique $\mathbf{d}_2 = \mathrm{dual}(\mathbf{d}_1)$ which is a $\mathfrak{u}_2$-free rectangle such that:

$$\alpha_{\mathbf{d}_2} = \beta_{\mathbf{d}_1}, \beta_{\mathbf{d}_2} = \alpha_{\mathbf{d}_1}, M^{\mathbf{d}_2}_{i,j} = M^{\mathbf{d}_1}_{j,i}$$

$$\mathbf{I}^{\mathbf{d}_2}_{i,j} = \mathbf{J}^{\mathbf{d}_1}_{j,i} \text{ and } \mathbf{J}^{\mathbf{d}_2}_{i,j} = \mathbf{I}^{\mathbf{d}_1}_{i,j}.$$

1.8 Definition. 1) We say τ is a weak $\mathfrak{u}$-sub-vocabulary <u>when</u>:

(a) $\tau \subseteq \tau_{\mathfrak{u}} = \tau_{\mathfrak{K}_{\mathfrak{u}}}$ except that $=_\tau$ is, in $\tau_{\mathfrak{u}}$, a two-place predicate such that for every $M \in \mathfrak{K}_{\mathfrak{u}}$ hence even $M \in \mathfrak{K}^{\mathrm{up}}_{\mathfrak{u}}$, the relation $=^M_\tau$ is an equivalence relation on $\mathrm{Dom}(=^M_\tau) = \{a : a =_\tau b \vee b =_\tau a$ for some $b \in M\}$ and is a congruence relation for all $R^M \restriction \mathrm{Dom}(=^M_\tau), F^M \restriction \mathrm{Dom}(=^M_\tau)$ for $R, F \in \tau$ and F^M maps[9] the set $\mathrm{Dom}(=^M_\tau)$ into itself for any function symbol $F \in \tau$.

So
1A) For $M \in K^{\mathrm{up}}_{\mathfrak{u}}$, the model $M^{[\tau]}$ is defined naturally, e.g. with universe $\mathrm{Dom}(=^M_\tau)/=^M_\tau$ and K^τ, K^τ_μ are defined accordingly. Let $M_1 \cong_\tau M_2$ means $M^{[\tau]}_1 \cong M^{[\tau]}_2$.
1B) Let $\dot{I}_\tau(\lambda, K^{\mathrm{up}}_{\mathfrak{u}}) = \{M^{[\tau]}/\cong : M \in K^{\mathfrak{u}}_\lambda$ and $M^{[\tau]}$ has cardinality $\lambda\}$.

[9]we may better ask less: for $F \in \tau$ a function symbol letting $n =$ arity$_\tau(F)$, so $F^{M^{[\tau]}}$ is a function with n-place from $\mathrm{Dom}(=^M_\tau)/=^M_\tau$ to itself and F^M is $\{(a_0, a_1, \dots, a_n) : (a_0/=^M = F(a_1/=^M), \dots, a_n/=^M)\}$, i.e. the graph of $M^{[\tau]}$, so we treat F as an (arity$_\tau(F)+1$)-place predicate; neither real change nor a real gain

1C) We say that τ is a strong $\mathfrak{u}$-sub-vocabulary when we have clause (a) from above and[10]

> (b) if $(M, N, \mathbf{I}) \in \mathrm{FR}_1^+$ then for some $c \in \mathbf{I} \cap \mathrm{Dom}(=_\tau^N)$ we have $N \models$ "$\neg(c =_\tau d)$" for every $d \in M$
>
> (c) if $(M_1, N_1, \mathbf{I}_1) \le_1 (M_2, N_2, \mathbf{I}_2)$ and $c \in \mathbf{I}_1$ is as in clause (b) for $(M_1, N_1, \mathbf{I}_1)$ then $c \in \mathbf{I}_2$ is as in (b) for $(M_2, N_2, \mathbf{I}_2)$.

2) We say that N_1, N_2 are τ-isomorphic over $\langle M_i : i < \alpha\rangle$ when: $\ell \in \{1,2\} \wedge i < \alpha \Rightarrow M_i \le_{\mathfrak{u}} N_\ell$ and there is a τ-isomorphism f of N_1 onto N_2 over $\bigcup\limits_{i<\alpha} M_i$ which means: f is an isomorphism from $N_1^{[\tau]}$ onto $N_2^{[\tau]}$ which is the identity on the universe of $M_i^{[\tau]}$ for each $i < \alpha$.
2A) In part (2), if $\alpha = 1$ we may write M_0 instead $\langle M_i : i < 1\rangle$ and we can replace M_0 by a set $\subseteq N_1 \cap N_2$. If $\alpha = 0$ we may omit "over $\bar{M}$".
3) We say that N_1, N_2 are τ-incompatible extensions of $\langle M_i : i < \alpha\rangle$ when:

> (a) $M_i \le_{\mathfrak{u}} N_\ell$ for $i < \alpha, \ell = 1,2$
>
> (b) if $N_\ell \le_{\mathfrak{u}} N'_\ell$ for $\ell = 1,2$ then N'_1, N'_2 are not τ-isomorphic over $\langle M_i : i < \alpha\rangle$.

4) We say that N_2^1, N_2^2 are τ-incompatible (disjoint) amalgamations of N_1, M_2 over M_1 when ($N_1 \cap M_2 = M_1$ and):

> (a) $M_1 \le_{\mathfrak{u}} N_1 \le_{\mathfrak{u}} N_2^\ell$ and $M_1 \le_{\mathfrak{u}} M_2 \le_{\mathfrak{u}} N_2^\ell$ for $\ell = 1,2$ (equivalently $M_1 \le_{\mathfrak{u}} N_1 \le_{\mathfrak{u}} N_2^\ell, M_1 \le_{\mathfrak{u}} M_2 \le_{\mathfrak{u}} N_2^\ell$)
>
> (b) if $N_2^\ell \le_{\mathfrak{u}} N_2^{\ell,*}$ for $\ell = 1,2$ then $(N_2^{1,*})^{[\tau]}, (N_2^{2,*})^{[\tau]}$ are not τ-isomorphic over $M_2 \cup N_1$, i.e. over $M_2^{[\tau]} \cup N_1^{[\tau]}$.

5) We say τ is a $\mathfrak{K}$-sub-vocabulary or K-sub-vocabulary when clause (a) of part (1) holds replacing $K_{\mathfrak{u}}$ by K; similarly in parts (1A),(1B).

1.9 Observation. Concerning 1.8(1B) we may be careless in checking the last condition, $= \lambda$, i.e. $\le \lambda$ usually suffice, because if $|\{M^{[\tau]}/\cong : M \in K_\lambda, \|M^{[\tau]}\| < \lambda\}| < \mu$ then in proving $\dot{I}_\tau(\lambda, K_{\mathfrak{u}}^{\mathrm{up}}) \ge \mu$ we may omit it.

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[10] note that it is important for us that the model we shall construct will be of cardinality ∂^+; this clause will ensure that the approximations will be of cardinality ∂ for $\alpha < \partial^+$ large enough and the final model (i.e. for $\alpha = \partial^+$) will be of cardinality ∂^+. This is the reason for a preference to $\le_1$, however there is no real harm in demanding clauses (b) + (c) for $\ell = 2$, too. But see 1.9, i.e. if $|\tau| \le \partial$ and we get $\mu > 2^\partial$ pairwise non-isomorphic models of cardinality $\le \partial^+$, clearly only few (i.e. $\le 2^\partial$) of them have cardinal $< \partial^+$; so this problem is not serious to begin with.

Remark. 1) But we give also remedies by FR^+_ℓ, i.e., clause (c) of 1.8(1).
2) We also give reminders in the phrasing of the coding properties.
3) If $|\tau| < \lambda$ and $2^{<\lambda} < \lambda$ the demand in 1.9 holds.

Proof. Should be clear. $\square_{1.9}$

1.10 Definition. 1) For any $(<\partial)$-a.e.c. $\mathfrak{K}$ let $\mathfrak{K}'$ be the $(<\partial)$-a.e.c. defined like $\mathfrak{K}$ only adding the two-place predicate $=_\tau$, demanding it to be a congruence relation, i.e.

- (a) $\tau' = \tau(\mathfrak{K}') = \tau \cup \{=_\tau\}$ where $\tau = \tau(\mathfrak{K})$
- (b) $K' = \{M : M$ is a τ'-model, $=^M_\tau$ is a congruence relation and $M/=^M_\tau$ belongs to κ and $\|M\| < \partial\}$
- (c) $M \le_{\mathfrak{K}'} N$ <u>iff</u> $M \subseteq N$ and the following function is a $\le_{\mathfrak{K}}$-embedding of $M/=^M_\tau$ into $N/=^N_\tau$: $f(a/=^M_\tau) = a/=^N_\tau$

(see Definition 1.8(1)).
1A) Similarly for $\mathfrak{K}$ an a.e.c. or a λ-a.e.c.
2) This is a special case of Definition 1.8.
3) We can interpret $M \in K$ as $M' \in K'$ just letting $M' \restriction \tau = M, =^{M'}_\tau$ is equality on $|M|$.
4) A model $M' \in \mathfrak{K}$ is called $=_\tau$-full when $a \in M' \Rightarrow \|M'\| = |\{b \in M' : M' \models a =_\tau b\}|$.
5) A model $M' \in \mathfrak{K}$ is called $(\lambda, =_\tau)$-full <u>when</u> $a \in M' \Rightarrow \lambda \le |\{b \in M' : M' \models$ "$a =_\tau b$"$\}|$.
6) A model M' is called $=_\tau$-fuller <u>when</u> it is $=_\tau$-full and $\|M'\|$ is the cardinality of $M'/=^{M'}_\tau$.

1.11 Claim. *Assume $\mathfrak{K}$ is*[11] *a $(<\partial)$-a.e.c. and*[12] *$\mathfrak{K}'$ is from 1.10 and $\lambda < \partial$.*
0) $\mathfrak{K}'_\lambda$ is a λ-a.e.c.
1) If $M', N' \in K'_\lambda$ <u>then</u> $(M'/=^{M'}_\tau) \in K_{\le\lambda}$ and $(N'/=^{N'}_\tau) \in K_{\le\lambda}$ and if in addition $M' \le_{\mathfrak{K}'_\lambda} N'$ <u>then</u> (up to identifying $a/=^{M'}_\tau$ with $q/=^{N'}_\tau$) we have $(M'/=^{M'}_\tau) \le_{\mathfrak{K}_{\le\lambda}} (N'/=^{N'}_\tau)$, i.e. $(M')^{[\tau]} \le_{\mathfrak{K}} (N')^{[\tau]}$.
2) If $M' \subseteq N'$ are $\tau'_{\mathfrak{K}}$-models of cardinality λ and $=^{M'}_\tau, =^{N'}_\tau$ are congruence relation on $M' \restriction \tau_{\mathfrak{K}}, N' \restriction \tau_{\mathfrak{K}}$ respectively, <u>then</u>

[11] We can use $\mathfrak{K}$ is an a.e.c. and have similar results.

[12] now pedantically $\mathfrak{K}$ may be both a $(<\partial_1)$-a.e.c. and a $(<\partial_2)$-a.e.c., e.g. if $\partial_2 = \partial_1^+, K_{\partial_1} = \emptyset$, so really ∂ should be given

(a) $M' \in K'_{<\partial}$ *iff* $(M'/=^{M'}_\tau) \in K_{<\partial}$

(b) $M' \le_{\mathfrak{K}'_{<\partial}} N'$ *iff* $(M'/=^{M'}_\tau) \le_{\mathfrak{K}_{<\partial}} (N'/=^{N'}_\tau)$; *pedantically*[13] $(M'/=^{N'}_\tau) \le_{\mathfrak{K}_\lambda} (N'/=^{N'}_\tau)$ *or make the natural identification*

(c) *if* M', N' *are* $=_\tau$*-fuller then in clauses (a),(b) we can replace* "$\le \lambda$" *by* "$= \lambda$"

(d) *if* $K_{<\lambda} = \emptyset$ *then in clauses (a),(b) we can replace* "$\le \lambda$" *by* "$= \lambda$".

3) $\mathfrak{K}'$ *has disjoint amalgamation* *if* $\mathfrak{K}$ *has amalgamation.*
4) $K_\lambda \subseteq K'_\lambda$ *and* $\le_{\mathfrak{K}_\lambda} = \le_{\mathfrak{K}'_\lambda} \restriction K_\lambda$.
5) $(\mathfrak{K}^{\rm up}_{<\mu})' = (\mathfrak{K}'_{<\mu})^{\rm up}$ *for any* μ *so we call it* $K'_{<\mu}$ *and* $\mathfrak{K}'_\mu = (\mathfrak{K}'_{<\mu^+})_\mu$.
6) For every μ,

(a) $\dot{I}(\mu, K) = |\{M'/\cong : M' \in K'_\mu$ *is* $=_\tau$*-fuller*$\}|$

(b) $K_\mu = \{M'/=^{M'}_\tau : M' \in K'_\mu$ *is* μ*-fuller*$\}$ *under the natural identification*

(c) $K_{\le\mu} = \{M/=^M_\tau : M \in K'_\mu\}$ *under the natural identification*

(d) *if* $M', N^1 \in K'_\mu$ *are* μ*-full* then $M' \cong N' \Leftrightarrow (M'/=^{M'}_\tau) \cong (N'/=^{N'}_\tau)$.

Proof. Straight. $\square_{1.11}$

1.12 Exercise: Assume $\mathfrak{K}, \mathfrak{K}'$ are as in 1.10.
1) If $\lambda \ge |\tau_{\mathfrak{K}}|$ and $2^\lambda < 2^{\lambda^+}$ then $\dot{I}(\lambda^+, \mathfrak{K}) + 2^\lambda = \dot{I}(\lambda^+, \mathfrak{K}') + 2^\lambda$, so if $\dot{I}(\lambda^+, \mathfrak{K}) > 2^\lambda$ or $\dot{I}(\lambda^+, \mathfrak{K}') > 2^\lambda$ then they are equal.
2) If $\lambda > |\tau_{\mathfrak{K}}|$ and $2^{<\lambda} < 2^\lambda$ then $\dot{I}(\lambda, \mathfrak{K}) + 2^{<\lambda} = \dot{I}(\lambda^+, \mathfrak{K}') + 2^{<\lambda}$ (and as above).

Remark. Most of our examples satisfies montonicity, see below.
But not so $\mathrm{FR}_1, \le_1$ in §4(C).

1.13 Exercise: Let $\mathfrak{u}$ be a nice construction framework, as usual.
1) [Definition] We say $\mathfrak{u}$ satisfies $(\mathrm{E})_\ell(\mathrm{e})$, monotonicity, when:

$(E)_\ell(e)$ if $(M, N, \mathbf{J}) \in \mathrm{FR}^\ell_{\mathfrak{u}}$ and $N \le_{\mathfrak{u}} N'$ then $(M, N, \mathbf{J}) \le^\ell_{\mathfrak{u}} (M, N, \mathbf{J}') \in \mathrm{FR}^\ell_{\mathfrak{u}}$.

1A) Let (E)(e) mean $(\mathrm{E})_1(\mathrm{e})$ + $(\mathrm{E})_2(\mathrm{e})$.
2) [Claim] Assume $\mathfrak{u}$ has monotonicity.

[13] or define when f is a $\le_{\mathfrak{K}'_{<\partial}}$-embedding of M^1 into N'

Assume $\mathbf{d}$ is a $\mathfrak{u}$-free (α_2, β_2)-rectangle, h_1, h_2 is an increasing continuous function from $\alpha_1 + 1, \beta_1 + 1$ into $\alpha_2 + 1, \beta_2 + 1$ respectively. Then $\mathbf{d}'$ is a $\mathfrak{u}$-free rectangle where we define $\mathbf{d}'$ by:

(a) $\alpha(\mathbf{d}') = \alpha_1, \beta(\mathbf{d}') = \beta_1$

(b) $M^{\mathbf{d}'}_{i,j} = M^{\mathbf{d}}_{h_1(i),h_2(1)}$ if $i \le \alpha_1, j \le \beta_1$

(c) $\mathbf{J}^{\mathbf{d}'}_{i,j} = \mathbf{J}^{\mathbf{d}}_{h_1(i),h_2(j)}$ for $i \le \alpha_1, j < \beta_1$

(d) $\mathbf{J}^{\mathbf{d}'}_{i,j} = \mathbf{J}^{\mathbf{d}}_{h_1(i),h_2(j)}$ for $i < \alpha_1, j \le \beta_1$.

3) [Claim] Phrase and prove the parallel of part (2) for $\mathfrak{u}$-free triangles.

1.14 Observation. Assume $\mathfrak{u}$ is a nice construction framework except that we omit clauses $(\mathrm{D})_\ell(\mathrm{e})$ + $(\mathrm{E})_\ell(\mathrm{d})$ for $\ell = 1, 2$ but satisfying Claim 1.3. We can show that $\mathfrak{u}'$ is a nice construction framework where we define $\mathfrak{u}'$ like $\mathfrak{u}$ but, for $\ell = 1, 2$, we replace $\mathrm{FR}_\ell, \le_\ell$ by $\mathrm{FR}'_\ell, \le'_\ell$ defined as follows:

(a) $\mathrm{FR}'_\ell = \{(M_1, M_2, \mathbf{J}) : (M_1, M_2, \mathbf{J}) \in \mathrm{FR}_\ell$ or $M_1 \le_{\mathfrak{u}} M_2 \in K_{<\partial}$ and $\mathbf{J} = \emptyset\}$

(b) $\le'_\ell = \{((M_1, N_1, \mathbf{J}'), (M_2, N_2, \mathbf{J}'')) : (M_1, N_1, \mathbf{J}') \le^\ell_{\mathfrak{u}} (M_2, N_2, \mathbf{J}'')$ <u>or</u> $M_1 \le_{\mathfrak{u}} N_1, \mathbf{J}' = \emptyset, M_1 \le_{\mathfrak{u}} M_2, N_1 \le_{\mathfrak{u}} N_2, N_1 \cap M_2 = M_1$ and $(M_2, N_2, \mathbf{J}'') \in \mathrm{FR}_\ell$ <u>or</u> $M_1 \le_{\mathfrak{u}} N_1 \le_{\mathfrak{u}} N_2, M_1 \le_{\mathfrak{u}} M_2 \le_{\mathfrak{u}} N_2, N_1 \cap M_2 = M_1$ and $\mathbf{J}' = \emptyset = \mathbf{J}''\}$.

Proof. Clauses (A),(B),(C) does not change, most subclauses of $(\mathrm{D})_\ell$(a),(b),(d),$(\mathrm{E})_\ell$(a),(b) hold by the parallel for $\mathfrak{u}$ and the choice of $\mathrm{FR}'_\ell, \le'_\ell$; clauses $(\mathrm{D})_\ell(\mathrm{e})$ and $(\mathrm{E})_\ell(\mathrm{d})$ holds by the choice of $(\mathrm{FR}'_\ell, \le'_\ell)$; and clause (F) holds by clause (F) for $\mathfrak{u}$ and Claim 1.3. Lastly

<u>Condition $(\mathrm{E})_\ell(\mathrm{c})$</u>:

So assume $\langle (M_i, N_i, \mathbf{J}_i) : i < \delta \rangle$ be increasing continuous, where δ is a limit ordinal; and let $(M_\delta, N_\delta, \mathbf{J}_\delta) = (\cup\{M_i : i < \delta\}, \cup\{N_i : i < \delta\}, \cup\{\mathbf{J}_i : i < \delta\})$.

First, assume $i < \delta \Rightarrow \mathbf{J}_i = \emptyset$ hence $\mathbf{J}_\delta = \emptyset$ and the desired conclusion holds trivially (by the properties of a.e.c. and our definition of $\mathfrak{u}'$).

Second, assume $i < \delta \not\Rightarrow \mathbf{J}_i \ne \emptyset$ hence $j := \min\{i : \mathbf{J}_i \ne \emptyset\}$ is well defined and let $\delta' = \delta - j$, it is a limit ordinal. Now use the "$\mathfrak{u}$ satisfies the Condition $(\mathrm{E})_\ell(\mathrm{c})$" for the sequence $\langle (M_{j+i}, N_{j+i}, \mathbf{J}_{j+i} : i < \delta' \rangle$ and $\le^\ell_{\mathfrak{u}}$ being transitive.

$\square_{1.14}$

* * *

Now we define the approximations of size ∂; note that the notation $\le_{\mathrm{qt}}$ and the others below hint that they are quasi orders, this will be justified later in 1.19(2), <u>but</u> not concerning $\le^{\mathrm{at}}_{\mathfrak{u}}$. On the existence of canonical limits see 1.19(4).

1.15 Definition. 1) We let $K^{\rm qt}_\partial = K^{\rm qt}_{\mathfrak u}$ be the class of triples $(\bar M, \bar{\mathbf J}, \mathbf f)$ such that

- (a) $\bar M = \langle M_\alpha : \alpha < \partial\rangle$ is $\le_{\mathfrak u}$-increasing continuous, so $M_\alpha \in K_{\mathfrak u}$ $(= K^{\mathfrak u}_{<\partial})$
- (b) $\bar{\mathbf J} = \langle \mathbf J_\alpha : \alpha < \partial\rangle$
- (c) $\mathbf f \in {}^\partial\partial$
- (d) $(M_\alpha, M_{\alpha+1}, \mathbf J_\alpha) \in$ FR$_2$ for $\alpha < \partial$.

1A) We call $(\bar M, \bar{\mathbf J}, \mathbf f) \in K^{\rm qt}_{\mathfrak u}$ non-trivial <u>if</u> for stationarily many $\delta < \partial$ for some $i < \mathbf f(\delta)$ we have $\mathbf J_{\delta+i} \ne \emptyset$ that is $(M_{\delta+i}, M_{\delta+i+1}, \mathbf J_{\delta+i}) \in$ FR$^+_2$.
1B) If $\mathscr D$ is a normal filter on ∂ let $K^{\rm qt}_{\mathscr D} = K^{\rm qt}_{\mathfrak u,\mathscr D}$ be the class of triples $(\bar M, \bar J, \mathbf f) \in K^{\rm qt}_{\mathfrak u}$ such that

- (e) $\{\delta < \partial : \mathbf f(\delta) = 0\} \in \mathscr D$.

1C) When we have $(\bar M^x, \bar{\mathbf J}^x, \mathbf f^x)$ then $M^x_\alpha, \mathbf J^x_\alpha$ for $\alpha < \partial$ has the obvious meaning and M^x_∂ or just M^x is $\cup\{M^x_\alpha : \alpha < \partial\}$
2) We define the two-place relation $\le_{\rm qt} = \le^{\rm qt}_{\mathfrak u}$ on $K^{\rm qt}_{\mathfrak u}$ as follows: $(\bar M^1, \bar{\mathbf J}^1, \mathbf f^1) \le^{\rm qt}_{\mathfrak u} (\bar M^2, \bar{\mathbf J}^2, \mathbf f^2)$ <u>if</u> they are equal (and $\in K^{\rm qt}_{\mathfrak u}$) or for some club E of ∂ (a witness) we have:

- (a) $(\bar M^k, \bar{\mathbf J}^k, \mathbf f^k) \in K^{\rm qt}_{\mathfrak u}$ for $k = 1, 2$
- (b) $\delta \in E \Rightarrow \mathbf f^1(\delta) \le \mathbf f^2(\delta)$
- (c) $\delta \in E\ \&\ i \le \mathbf f^1(\delta) \Rightarrow M^1_{\delta+i} \le_{\mathfrak u} M^2_{\delta+i}$
- (d) $\delta \in E\ \&\ i < \mathbf f^1(\delta) \Rightarrow (M^1_{\delta+i}, M^1_{\delta+i+1}, \mathbf J^1_{\delta+i}) \le_2 (M^2_{\delta+i}, M^2_{\delta+i+1}, \mathbf J^2_{\delta+i})$
- (e) $\delta \in E\ \&\ i \le \mathbf f^1(\delta) \Rightarrow M^2_{\delta+i} \cap \bigcup\limits_{\alpha<\partial} M^1_\alpha = M^1_{\delta+i}$, disjointness.

3) We define the two place relation $\le_{\rm at} = \le^{\rm at}_{\mathfrak u}$ on $K^{\rm qt}_{\mathfrak u}$: $(\bar M^1, \bar{\mathbf J}^1, \mathbf f^1) \le^{\rm at}_{\mathfrak u} (\bar M^2, \bar{\mathbf J}^2, \mathbf f^2)$ <u>if</u> for some club E of ∂ and $\bar{\mathbf I}$ (the witnesses) we have (a)-(e) as in part (2) and

- (f) $\bar{\mathbf I} = \langle \mathbf I_\alpha : \alpha < \partial\rangle$ and $\langle (M^1_\alpha, M^2_\alpha, \mathbf I_\alpha) : \alpha \in \cup\{[\delta, \delta + \mathbf f^1(\delta)] : \delta \in E\}\rangle$ is $\le_1$-increasing continuous, so we may use $\langle \mathbf I_\alpha : \alpha \in \cup\{[\delta, \delta + \mathbf f^1(\delta)] : \delta \in E\rangle$ only.

3A) We say $(\bar M^1, \bar{\mathbf J}^1, \mathbf f^1), (\bar M^2, \bar{\mathbf J}^2, \mathbf f^2)$ are equivalent when for a club of $\delta < \partial$ we have $\mathbf f^1(\delta) = \mathbf f^2(\delta)$ and $i \le \mathbf f^1(\delta) \Rightarrow M^1_{\delta+i} = M^2_{\delta+i}$ and $i < \mathbf f^1(\delta) \Rightarrow \mathbf J^1_{\delta+i} = \mathbf J^2_{\delta+i}$.
3B) Let $(\bar M^1, \bar{\mathbf J}^1, \mathbf f^1) <^{\rm at}_{\mathfrak u} (\bar M^2, \bar{\mathbf J}^2, \mathbf f^2)$ mean that in part (3) in addition

- (g) for some $\alpha \in \cup\{[\delta, \delta + \mathbf f(\delta)] : \delta \in E\}$ the triple $(M^1_\alpha, M^2_\alpha, \mathbf I_\alpha)$ belongs to FR$^+_1$.

4) We say $(\bar{M}^\delta, \bar{\mathbf{J}}^\delta, \mathbf{f}^\delta)$ is a canonical limit of $\langle (\bar{M}^\alpha, \bar{\mathbf{J}}^\alpha, \mathbf{f}^\alpha) : \alpha < \delta \rangle$ <u>when</u>:

- (a) $\delta < \partial^+$
- (b) $\alpha < \beta < \delta \Rightarrow (\bar{M}^\alpha, \bar{\mathbf{J}}^\alpha, \mathbf{f}^\alpha) \leq^{\mathrm{qt}}_{\mathfrak{u}} (\bar{M}^\beta, \bar{\mathbf{J}}^\beta, \mathbf{f}^\beta)$
- (c) for some increasing continuous sequence $\langle \alpha_\varepsilon : \varepsilon < \operatorname{cf}(\delta) \rangle$ of ordinals with limit δ we have:

<u>Case 1</u>: $\operatorname{cf}(\delta) < \partial$.
For some club E of ∂ we have:

- (α) $\zeta \in E \Rightarrow \mathbf{f}^\delta(\zeta) = \sup\{\mathbf{f}^{\alpha_\varepsilon}(\zeta) : \varepsilon < \operatorname{cf}(\delta)\}$;
- (β) $\zeta \in E \ \& \ \xi < \operatorname{cf}(\delta) \ \& \ i \leq \mathbf{f}^{\alpha_\xi}(\zeta)$ implies that $M^\delta_{\zeta+i} = \cup\{M^{\alpha_\varepsilon}_{\zeta+i} : \varepsilon < \operatorname{cf}(\delta)$ satisfies $\varepsilon \geq \xi\}$;
- (γ) $\zeta \in E \ \& \ \xi < \operatorname{cf}(\delta) \ \& \ i < \mathbf{f}^\xi(\zeta)$ implies that $\mathbf{J}^\delta_{\zeta+i} = \cup\{\mathbf{J}^{\alpha_\varepsilon}_{\zeta+i} : \varepsilon < \operatorname{cf}(\delta)$ satisfies $\varepsilon \geq \xi\}$
- (δ) if $\zeta \in E$ and $j = \mathbf{f}^\delta(\zeta) > \mathbf{f}^{\alpha_\varepsilon}(\zeta)$ for every $\varepsilon < \operatorname{cf}(\delta)$ then $M^\delta_{\zeta+j} = \cup\{M^\delta_{\zeta+i} : i < j\}$.

<u>Case 2</u>: $\operatorname{cf}(\delta) = \partial$.
Similarly, using diagonal unions.
4A) We say $\langle (\bar{M}^\alpha, \bar{\mathbf{J}}^\alpha, \mathbf{f}^\alpha) : \alpha < \alpha(*) \rangle$ is $\leq^{\mathrm{qt}}_{\mathfrak{u}}$-increasing continuous <u>when</u> it is $\leq^{\mathrm{qt}}_{\mathfrak{u}}$-increasing and for every limit ordinal $\delta < \alpha(*)$, the triple $(\bar{M}^\delta, \bar{\mathbf{J}}^\delta, \mathbf{f}^\delta)$ is a canonical limit of $\langle (\bar{M}^\alpha, \bar{\mathbf{J}}^\alpha, \mathbf{f}^\alpha) : \alpha < \delta \rangle$.
5) We define the relation $\leq_{\mathrm{qs}} = \leq^{\mathrm{qs}}_{\mathfrak{u}}$ on $K^{\mathrm{qt}}_{\mathfrak{u}}$ by: $(\bar{M}', \bar{\mathbf{J}}', \mathbf{f}') \leq_{\mathrm{qs}} (\bar{M}'', \bar{\mathbf{J}}'', \mathbf{f}'')$ <u>if</u> there is a $\leq^{\mathrm{at}}_{\mathfrak{u}}$-tower $\langle (\bar{M}^\alpha, \bar{\mathbf{J}}^\alpha, \mathbf{f}^\alpha) : \alpha \leq \alpha(*) \rangle$ witnessing it, meaning that is is a sequence such that:

- (a) the sequence is $\leq^{\mathrm{qt}}_{\mathfrak{u}}$-increasing of length $\alpha(*) + 1 < \partial^+$
- (b) $(\bar{M}^0, \bar{\mathbf{J}}^0, \mathbf{f}^0) = (\bar{M}', \bar{\mathbf{J}}', \mathbf{f}')$
- (c) $(\bar{M}^{\alpha(*)}, \bar{\mathbf{J}}^{\alpha(*)}, \mathbf{f}^{\alpha(*)}) = (\bar{M}'', \bar{\mathbf{J}}'', \mathbf{f}'')$
- (d) $(\bar{M}^\alpha, \bar{\mathbf{J}}^\alpha, \mathbf{f}^\alpha) \leq^{\mathrm{at}}_{\mathfrak{u}} (\bar{M}^{\alpha+1}, \bar{\mathbf{J}}^{\alpha+1}, \mathbf{f}^{\alpha+1})$ for $\alpha < \alpha(*)$
- (e) if $\delta \leq \alpha(*)$ is a limit ordinal then $(\bar{M}^\delta, \bar{\mathbf{J}}^\delta, \mathbf{f}^\delta)$ is a canonical limit of $\langle (\bar{M}^\alpha, \bar{\mathbf{J}}^\alpha, \mathbf{f}^\alpha) : \alpha < \delta \rangle$.

5A) Let $<_{\mathrm{qs}} = <^{\mathrm{qs}}_{\mathfrak{u}}$ be defined similarly but for at least one $\alpha < \alpha(*)$ we have $(\bar{M}^\alpha, \bar{\mathbf{J}}^\alpha, \mathbf{f}^\alpha) <^{\mathrm{at}}_{\mathfrak{u}} (\bar{M}^{\alpha+1}, \bar{\mathbf{J}}^{\alpha+1}, \mathbf{f}^{\alpha+1})$.
5B) Let $\leq_{\mathrm{qr}} = \leq^{\mathrm{qr}}_{\mathfrak{u}}$ be defined as in part (5) but in clause (d) we use $<^{\mathrm{at}}_{\mathfrak{u}}$. Similarly for $<_{\mathrm{qr}} = <^{\mathrm{qr}}_{\mathfrak{u}}$, i.e. when $\alpha(*) > 0$.
6) We say that $\langle (\bar{M}^\alpha, \bar{\mathbf{J}}^\alpha, \mathbf{f}^\alpha) : \alpha < \alpha(*) \rangle$ is $\leq^{\mathrm{qs}}_{\mathfrak{u}}$-increasing continuous <u>when</u> it is $\leq^{\mathrm{qs}}_{\mathfrak{u}}$-increasing and clause (e) of part (5) holds. Similarly for $\leq^{\mathrm{qr}}_{\mathfrak{u}}$.

Some obvious properties are (see more in Observation 1.19).

1.16 Observation. 1) $(\bar M^1, \bar{\mathbf J}^1, \mathbf f^1) \le^{\rm at}_{\mathfrak u} (\bar M^2, \bar{\mathbf J}^2, \mathbf f^2)$ <u>iff</u> for some club E of ∂ and sequence $\bar{\mathbf I} = \langle \mathbf I_\alpha : \alpha \in \cup\{[\delta, \delta + \mathbf f^1(\delta)] : \alpha \in E\}\rangle$ we have clause (a),(b),(c) of Definition 1.15(2) and

- $(f)'$ $(\langle M^1_{\delta+i} : i \le \mathbf f^1(\delta)\rangle, \langle M^2_{\delta+i} : i \le \mathbf f^1(\delta)\rangle, \langle \mathbf J^1_{\delta+i} : i < \mathbf f^1(\delta)\rangle, \langle \mathbf J^2_{\delta+i} : i < \mathbf f^1(\delta)\rangle, \langle \mathbf I_{\delta+i} : i \le \mathbf f^1(\delta)\rangle)$ is a $\mathfrak u$-free $(\mathbf f^1(\delta), 1)$-rectangle
- $(f)'_1$ if $\delta_1 < \delta_2$ are from E then $(M^1_{\delta_1+\mathbf f^1(\delta_1)}, M^2_{\delta_1+\mathbf f^1(\delta_1)}, \mathbf I_{\delta_1+\mathbf f^1(\delta_1)}) \le_2 (M^1_{\delta_2}, M^2_{\delta_2}, \mathbf I_{\delta_2})$.

2) The relation $\le^{\rm qt}_{\mathfrak u}, \le^{\rm at}_{\mathfrak u}, \le^{\rm qs}_{\mathfrak u}, \le^{\rm qr}_{\mathfrak u}$ and $\le^{\rm at}_{\mathfrak u}$ are preserved by equivalence, see Definition 1.15(3A) (and equivalence is an equivalence relation) and so are $<_{\rm at}, <_{\rm qt}, <_{\rm qs}$ $, <_{\rm qr}$.

Proof. Straightforward. $\square_{1.16}$

1.17 Remark. 1) In some of our applications it is natural to redefine the partial order $\le^{\rm qs}_{\mathfrak u}$ we use on $K^{\rm qt}_{\mathfrak u}$ as the closure of a more demanding relation.
2) If we demand $\mathrm{FR}_1 = \mathrm{FR}^+_1$ hence we omit clause $(D)_1(e), (E)_1(d)$ of Definition 1.2, really $<^{\rm at}_{\mathfrak u}$ is the same as $\le^{\rm at}_{\mathfrak u}$. In Definition 1.15(3) we can choose $\mathbf I_\alpha = \emptyset$, then we get $\le^{\rm qt}_{\mathfrak u}$. But even so we would like to be able to say "repeat §1 with the following modifications". If in Definition 1.15(5) clause (d) we use $<^{\rm at}_{\mathfrak u}$, i.e. use $<^{\rm qr}_{\mathfrak u}$, the difference below is small.
3) Note that below $K^{\mathfrak u,*}_\partial \subseteq K^{\rm up}_{\mathfrak u,\partial}$ and $K^{\mathfrak u,*}_{\partial^+} \subseteq K^{\rm up}_{\mathfrak u,\partial^+}$, see the definition below.
4) Should we use $\le^{\rm qs}_{\mathfrak u}$ or $\le^{\rm qr}_{\mathfrak u}$ (see Definition 1.15(5A),(5B))? So far it does not matter.

1.18 Definition. 1) $K^{\mathfrak u,*}_\partial = \{M : M = \cup\{M_\alpha : \alpha < \partial\}$ for some non-trivial $(\bar M, \bar{\mathbf J}, \mathbf f) \in K^{\rm qt}_{\mathfrak u}\}$, recalling Definition 1.15(1A).
2) $K^{\mathfrak u,*}_{\partial^+} = \{\bigcup\{M^\gamma : \gamma < \partial^+\} : \langle(\bar M^\gamma, \bar{\mathbf J}^\gamma, \mathbf f^\gamma) : \gamma < \partial^+\rangle$ is $\le^{\rm qs}_{\mathfrak u}$-increasing continuous and for unboundedly many $\gamma < \partial^+$ we have $(\bar M^\alpha, \bar{\mathbf J}^\alpha, \mathbf f^\alpha) <^{\rm qs}_{\mathfrak u} (\bar M^{\alpha+1}, \bar{\mathbf J}^{\alpha+1}, \mathbf f^{\alpha+1})$ and as usual $M^\gamma = \cup\{M^\gamma_\alpha : \alpha < \partial\}\}$.

1.19 Observation. 1) $K^{\rm qt}_{\mathfrak u} \ne \emptyset$; moreover it has non-trivial members.
2) The two-place relations $\le^{\rm qt}_{\mathfrak u}$ and $\le^{\rm qs}_{\mathfrak u}, \le^{\rm qr}_{\mathfrak u}$ are quasi orders and so are $<_{\rm qt}, <_{\rm qs}, <_{\rm qr}$ <u>but</u> not necessarily $\le_{\rm at}, <_{\rm at}$.
3) Assume $(\bar M^1, \bar{\mathbf J}^1, \mathbf f^1) \in K^{\rm qt}_{\mathfrak u}$ and $\alpha < \partial, (M^1_\alpha, M^2_\alpha, \mathbf I^*) \in \mathrm{FR}_1$ and $\mathbf f^1 \le_{\mathscr D_\partial} \mathbf f^2 \in {}^\partial\partial$ and $M^2_\alpha \cap M^1 = M^1_\alpha$. <u>Then</u> we can find $\bar M^2, \bar{\mathbf J}^2, E$ and $\bar{\mathbf I} = \langle \mathbf I_\alpha : \alpha \in \cup\{[\delta, \mathbf f^1(\delta)] : \delta \in E\}\rangle$ such that

(a) $(\bar M^1, \mathbf{J}^1, \mathbf{f}^1) \le^{\rm at}_{\mathfrak u} (\bar M^2, \bar{\mathbf{J}}^2, \mathbf{f}^2)$ as witnessed by $E, \bar{\mathbf{I}}$

(b) if $\beta \in \cup\{[\delta, \mathbf{f}^2(\delta)] : \delta \in E\}$ then $(M^1_\alpha, M^2_\alpha, \mathbf{I}^*) \le_1 (M^1_\beta, M^2_\beta, \mathbf{I}_\beta)$

(c) if $(M^1_\alpha, M^2_\alpha, \mathbf{I}^*) \in \mathrm{FR}^+_1$ then $(\bar M^1, \bar{\mathbf{J}}^1, \mathbf{f}^1) <^{\rm qs}_{\mathfrak u} (\bar M^2, \bar{\mathbf{J}}^2, \mathbf{f}^2)$ moreover $(\bar M^1, \bar{\mathbf{J}}^1, \mathbf{f}^1) <^{\rm at}_{\mathfrak u} (\bar M^2, \bar{\mathbf{J}}^2, \mathbf{f}^2)$.

4) If $\langle (\bar M^\alpha, \bar{\mathbf{J}}^\alpha, \mathbf{f}^\alpha) : \alpha < \delta\rangle$ is $\le^{\rm qt}_{\mathfrak u}$-increasing continuous (i.e. we use canonical limits) and δ is a limit ordinal $< \partial^+$, then it has a canonical limit $(\bar M^\delta, \bar{\mathbf{J}}^\delta, \mathbf{f}^\delta)$ which is unique up to equivalence (see 1.15(3A)). Similarly for $\le^{\rm qs}_{\mathfrak u}$ and $\le^{\rm qr}_{\mathfrak u}$.
5) $K^{\mathfrak u,*}_\partial, K^{\mathfrak u,*}_{\partial^+}$ are non-empty and included in $K^{\mathfrak u}$, in fact in $K^{\mathfrak u}_\partial, K^{\mathfrak u}_{\partial^+}$, respectively. Also if τ is a strong $\mathfrak u$-sub-vocabulary and $M \in K^{\mathfrak u,*}_\partial$ or $M \in K^{\mathfrak u,*}_{\partial^+}$ then $M^{[\tau]}$ has cardinality ∂ or ∂^+ respectively. If τ is a weak $\mathfrak u$-subvocabulary we get only $\le \partial, \le \partial^+$ respectively.

Proof. 1) We choose $M_i \in K_{\mathfrak u} = \mathfrak K_{<\partial}, \le_{\mathfrak u}$-increasing continuous with i as follows. For $i = 0$ use $\mathfrak K_{<\partial} \ne \emptyset$ by clause (C) of Definition 1.2. For i limit note that $\langle M_j : j < i\rangle$ is $\le_{\mathfrak K}$-increasing continuous and $j < i \Rightarrow M_j \in \mathfrak K_{\mathfrak u}$ hence $j < i \Rightarrow \|M_j\| < \partial$ but $i < \partial$ and ∂ is regular (by Definition 1.2, clause (B)) so $M_i := \cup\{M_j : j < i\}$ has cardinality $< \partial$ hence (by clause (C) of Definition 1.2) $M_i \in \mathfrak K_{\mathfrak u}$ and $j < i \Rightarrow M_j \le_{\mathfrak u} M_i$. For $i = j+1$ by clause $(D)_2(d)$ of Definition 1.2 there are $M_i, \mathbf{J}_j$ such that $(M_j, M_i, \mathbf{J}_j) \in \mathrm{FR}^+_2$. Choose $\mathbf{f} \in {}^\partial\partial$, e.g., $\mathbf{f}(\alpha) = 1$.
Now $(\langle M_i : i < \partial\rangle, \langle \mathbf{J}_i : i < \partial\rangle, \mathbf{f}) \in K^{\rm qt}_{\mathfrak u}$ is as required; moreover is non-trivial, see Definition 1.15(1A).
2) We first deal with $\le^{\rm qt}_{\mathfrak u}$.

Trivially $(\bar M, \bar{\mathbf{J}}, \mathbf{f}) \le^{\rm qt}_{\mathfrak u} (\bar M, \bar{\mathbf{J}}, \mathbf{f})$ for $(M, \bar{\mathbf{J}}, \mathbf{f}) \in K^{\rm qt}_{\mathfrak u}$.
[Why? It is witnessed by $E = \partial$ (as $(M_i, M_i, \emptyset) \in \mathrm{FR}_1$ by clause $(D)_1(e)$ of Definition 1.2 and $i < j < \partial \Rightarrow (M_i, M_i, \emptyset) \le_1 (M_j, M_j, \emptyset)$ by clause $(E)_1(d)$ of Definition 1.2.]
So assume $(\bar M^\ell, \bar{\mathbf{J}}^\ell, \mathbf{f}) \le^{\rm qt}_{\mathfrak u} (\bar M^{\ell+1}, \bar{\mathbf{J}}^{\ell+1}, \mathbf{f}^\ell)$ and let it be witnessed by E_ℓ for $\ell = 1, 2$. Let $E = E_1 \cap E_2$, it is a club of ∂. For every $\delta \in E$ by clause (b) of Definition 1.15(2) we have $\mathbf{f}^1(\delta) \le \mathbf{f}^2(\delta) \le \mathbf{f}^3(\delta)$ hence $\mathbf{f}^1(\delta) \le \mathbf{f}^3(\delta)$ and for $i \le \mathbf{f}^1(\delta)$ by clause (c) of Definition 1.15(2), clearly $M^1_{\mathbf{f}^1(\delta)+i} \le_{\mathfrak u} M^2_{\mathbf{f}^1(\delta)+i} \le_{\mathfrak u} M^3_{\mathbf{f}^1(\delta)+i}$ so as $\mathfrak K_{\mathfrak u}$ is a $(< \partial)$-a.c.e. we have $M^1_{\delta+i} \le_{\mathfrak u} M^3_{\delta+i}$. Similarly as $\le^2_{\mathfrak u}$ is a quasi order by clause (E)$_2$(a) of Definition 1.2 we have $\delta \in E$ & $i < \mathbf{f}^1(\delta) \Rightarrow (M^1_{\delta+i}, M^1_{\delta+i+1}, \mathbf{J}^1_{\delta+i}) \le^2_{\mathfrak u} (M^3_{\delta+i}, M^3_{\delta+i+1}, \mathbf{J}^3_{\delta+i})$ so clause (d) of Definition 1.15(2) holds.

Also if $\delta \in E$ and $i \le \mathbf{f}^1(\delta)$ then $M^2_{\delta+i} \cap (\cup\{M^1_\gamma : \gamma < \partial\}) = M^1_{\delta+i}$ and $M^3_{\delta+i} \cap (\cup\{M^2_\gamma : \gamma < \partial\}) = M^2_{\delta+i}$ hence $M^3_{\delta+i} \cap (\cup\{M^1_\gamma : \gamma < \partial\}) = M^1_{\delta+i}$, i.e. clause (e) there holds and clause (a) is trivial.

Together really $(\bar M^1, \bar{\mathbf{J}}^1, \mathbf{f}^1) \le^{\rm qt}_{\mathfrak u} (\bar M^3, \bar{\mathbf{J}}^3, \mathbf{f}^3)$.
So $\le^{\rm qt}_{\mathfrak u}$ is actually a quasi order. As for $\le^{\rm qs}_{\mathfrak u}$ and $\le^{\rm qr}_{\mathfrak u}$, this follows by the result on

$\leq^{\rm qt}_{\mathfrak u}$ and the definitions. Similarly for $<^{\rm qt}_{\mathfrak u}, <^{\rm qs}_{\mathfrak u}, <^{\rm qr}_{\mathfrak u}$.
3) By induction on $\beta \in [\alpha, \partial)$ we choose $(g_\beta, M^2_\beta, \mathbf{I}_\beta)$ and $\mathbf{J}_\beta$ (but $\mathbf{J}_\beta$ is chosen in the $(\beta+1)$-th step) such that

- (a) $M^2_\beta \in \mathfrak{K}_{\mathfrak u}$ is $\leq_{\mathfrak u}$-increasing continuous
- (b) g_β is a $\leq_{\mathfrak u}$-embedding of M^1_β into M^2_β, increasing and continuous with β
- (c) $(g_\beta(M^1_\beta), M^2_\beta, \mathbf{I}_\beta) \in$ ${\rm FR}^{\mathfrak u}_1$ is $\leq^1_{\mathfrak u}$-increasing and continuous
- (d) if $\beta = \gamma+1$ then $(g_\gamma(M^1_\gamma), g_\beta(M^1_\beta), g_\beta(\mathbf{J}^1_\beta)) \leq^2_{\mathfrak u} (M^2_\gamma, M^2_\beta, \mathbf{J}^2_\beta)$.

For $\beta = \alpha$ let $g_\beta =$ ${\rm id}_{M^1_\alpha}, M^2_\beta$ as given, $\mathbf{I}_\beta = \mathbf{I}^*$, so by the assumptions all is O.K.

For β limit use clause $(E)_1(c)$ of Definition 1.2.

For $\beta = \gamma+1$ use clause (F) of Definition 1.2. Having carried the induction, by renaming without loss of generality $g_\beta =$ ${\rm id}_{M^1_\beta}$ for $\beta < \partial$. So clearly we are done.
4),5) Easy, too. $\square_{1.19}$

1.20 Definition. 1) $K^{\rm qt}_\alpha = K^{\rm qt}_{\mathfrak u,\alpha}$ is defined as in Definition 1.15(1) above but $\alpha =$ ${\rm Dom}(\mathbf{f}) = \ell g(\bar{\mathbf{J}}) = \ell g(\bar M) - 1$, where $\alpha \leq \partial$.
2) $K^{\rm qt}_{<\alpha} = K^{\rm qt}_{\mathfrak u,<\alpha}, K^{\rm qt}_{\leq\alpha} = K^{\rm qt}_{\mathfrak u,\leq\alpha}$ are defined similarly.

* * *

<u>1.21 Discussion</u>: 1) Central here in Chapter VII are "τ-coding properties" meaning that they will help us in building $M \in K^{\mathfrak u}_{\partial^+}$, moreover in $K^{\mathfrak u,*}_{\partial^+}$ (or $K^{\mathfrak u,\mathfrak h}_{\partial^+}$, see below) such that we can code some subset of ∂^+ by the isomorphism type of $M^{[\tau]}$; that is during the construction, choosing $(\bar M^\alpha, \bar{\mathbf{J}}^\alpha, \mathbf{f}^\alpha) \in K^{\rm qt}_{\mathfrak u}$ which are $\leq^{\rm qs}_{\mathfrak u}$-increasing with $\alpha < \partial^+$, we shall have enough free decisions. This means that, arriving to the α-th triple we have continuations which are incompatible in some sense. This will be done in §2,§3.
2) The following definition will help phrase coding properties which holds just for "almost all" triples from $K^{\rm qt}_{\mathfrak u}$. Note that in the weak version of coding we have to preserve $\mathbf{f}(\delta) = 0$ for enough δ's.
3) In the applications we have in mind, $\partial = \lambda^+$, the set of $(\bar M, \bar{\mathbf{J}}, \mathbf{f}) \in K^{\rm qt}_{\mathfrak u}$ for which $M_{\lambda^+} = \cup\{M_i : i < \lambda^+\}$ is saturated above λ, is dense enough which for our purpose means that for almost every $(\bar M, \bar{\mathbf{J}}, \mathbf{f})$ this holds.
4) Central in our proof will be having "for almost all $(\bar M, \bar{\mathbf{J}}, \mathbf{f}) \in K^{\rm qt}_{\mathfrak u}$ in some sense, satisfies ....". The first version (almost$_3$, in 1.22(0)), is related to Definition 1.24.
5) The version of Definition 1.22 we shall use mostly in 1.22(3C), "almost$_2$...", which means that for some stationary $S \subseteq \partial$, we demand the sequences to "strictly S-obey $\mathfrak g$"; and from Definition 1.24 is 1.24(7), "$\{0,2\}$-almost".

1.22 Definition. 0) We say that "almost$_3$ every $(\bar{M}, \bar{\mathbf{J}}, \mathbf{f}) \in K^{\rm qt}_{\mathfrak{u}}$ satisfies Pr" when there is a function $\mathfrak{h}$ witnessing it which means:

$(*)_1$ $(\bar{M}^\delta, \bar{\mathbf{J}}^\delta, \mathbf{f}^\delta)$ satisfies Pr when: the sequence $\mathbf{x} = \langle (\bar{M}^\alpha, \bar{\mathbf{J}}^\alpha, \mathbf{f}^\alpha) : \alpha \le \delta \rangle$ is $\le^{\rm qs}_{\mathfrak{u}}$-increasing continuous ($\delta < \partial^+$ a limit ordinal, of course) and obeys $\mathfrak{h}$ which means that for some unbounded subset u of δ for every $\alpha \in u$ the sequence $\mathbf{x}$ does obey$_3$ or 3-obeys $\mathfrak{h}$ which means $(\bar{M}^{\alpha+1}, \bar{\mathbf{J}}^{\alpha+1}, \mathbf{f}^{\alpha+1}) = \mathfrak{h}((\bar{M}^\alpha, \bar{\mathbf{J}}^\alpha, \mathbf{f}^\alpha)$ (and for notational simplicity the universes of $M^\alpha_\partial, M^{\alpha+1}_\partial$ are sets of ordinals[14]); we may write obeys instead 3-obeys when this is clear from the context; also below

$(*)_2$ above $\mathbf{f}^\alpha(i) = 0 \Rightarrow \mathbf{f}^{\alpha+1}(i) = 0$ for a club of $i < \partial$ and $\{i : \mathbf{f}^\alpha(i) > 0\}$ is stationary for every $\alpha \le \delta$

$(*)_3$ $\mathfrak{h}$ has the domain and range implicit in $(*)_1 + (*)_2$

$(*)_4$ we shall restrict ourselves to a case where each of the models M^α_i above have universe $\subseteq \partial^+$, (or just be a set of ordinals) thus avoiding the problem of global choice; similarly below (e.g. in part (3)).

1) We say that the pair $((\bar{M}^1, \bar{\mathbf{J}}^1, \mathbf{f}^1), (\bar{M}^2, \bar{\mathbf{J}}^2, \mathbf{f}^2))$ does S-obey or S-obeys$_1$ the function $\mathfrak{g}$ (or $(\bar{M}^2, \bar{\mathbf{J}}^2, \mathbf{f}^2)$ does S-obeys or S-obeys$_1$ $\mathfrak{g}$ above $(\bar{M}^1, \bar{\mathbf{J}}, \mathbf{f}^1)$), when for some $\bar{\mathbf{I}}$ and E we have

(a) S is a stationary subset of ∂ and E is a club of ∂

(b) $(\bar{M}^1, \bar{\mathbf{J}}^1, \mathbf{f}^1) <^{\rm at}_{\mathfrak{u}} (\bar{M}^2, \bar{\mathbf{J}}^2, \mathbf{f}^2)$ as witnessed by E and $\bar{\mathbf{I}}$

(c) for stationarily many $\delta \in S$

- $\odot$ the triple $(\bar{M}^2 \restriction (\delta + \mathbf{f}^2(\delta) + 1), \bar{\mathbf{J}}^2 \restriction (\delta + \mathbf{f}^2(\delta)), \bar{\mathbf{I}} \restriction (\delta + \mathbf{f}^2(\delta)) + 1)$ is equal to, (in particular[15] $\mathfrak{g}$ is well defined in this case) $\mathfrak{g}(\bar{M}^1, \bar{\mathbf{J}}^1, \mathbf{f}^1, \bar{M}^2 \restriction (\delta + \mathbf{f}^1(\delta) + 1), \bar{\mathbf{J}}^2 \restriction (\delta + \mathbf{f}^1(\delta), \bar{\mathbf{I}} \restriction (\delta + \mathbf{f}^1(\delta) + 1), S)$
  or at least

- $\odot'$ for some $\gamma_1 \le \gamma_2$ from the interval $[\mathbf{f}^1(\delta), \mathbf{f}^2(\delta)]$, the triple $(\bar{M}^2 \restriction (\gamma_2 + 1), \bar{\mathbf{J}}_2 \restriction \gamma_2, \bar{\mathbf{I}} \restriction (\gamma_2 + 1))$ is equal to $\mathfrak{g}(\bar{M}^1, \bar{\mathbf{J}}^1, \mathbf{f}^1, \bar{M}^2 \restriction (\gamma_1 + 1)), \bar{\mathbf{J}}^2 \restriction \gamma_1, \bar{\mathbf{I}} \restriction (\gamma_1 + 1), S)$.

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[14] we may alternatively restrict yourself to models with universe $\subseteq \partial^+$ or use a universal choice function. Also if we use $\mathfrak{h}(\langle \bar{M}^\beta, \bar{\mathbf{J}}^\beta, \mathbf{f}^\beta) : \beta \le \alpha \rangle)$ the difference is minor: make the statement a little cumbersome and the checking a little easier. Presently we do not distinguish the two versions.

[15] alternatively we can demand (as in §9,§10) that: the universe of M^1_∂ and of M^2_∂ is an ordinal $< \partial^+$

1A) Saying "strictly S-obeys$_1$" mean that in clause (1)(c) we replace "stationarily many $\delta \in S$" by "every $\delta \in E \cap S$ (we can add the "strictly" in other places, too). Omitting S means for some stationary $S \subseteq \partial$; we may assume $\mathfrak{g}$ codes S and in this case we write $S = S_{\mathfrak{g}}$ and can omit S. In the end of clause (1)(c), if the resulting value does not depend on some of the objects written as arguments we may omit them. We may use $\bar{\mathfrak{g}} = \langle \mathfrak{g}_S : S \subseteq \partial \text{ stationary} \rangle$ and obeying $\bar{\mathfrak{g}}$ means obeying $\mathfrak{g}_S$ for some S (where $\mathfrak{g} = \mathfrak{g}_S \Rightarrow S_{\mathfrak{g}} = S$).
2) A $\le^{\rm qs}_{\mathfrak{u}}$-increasing continuous sequence $\langle (\bar{M}^\alpha, \bar{\mathbf{J}}^\alpha, \mathbf{f}^\alpha) : \alpha \le \delta \rangle$ obeys$_1$ or 1-obeys $\bar{\mathfrak{g}}$ when δ is a limit ordinal $< \partial^+$ and for some unbounded $u \subseteq \delta$ there is a sequence $\langle S_\alpha : \alpha \in u \cup \{\delta\} \rangle$ of stationary subsets of ∂ decreasing modulo $\mathscr{D}_\partial$ such that for each $\alpha \in u$, the pair $((\bar{M}^\alpha, \bar{\mathbf{J}}^\alpha, \mathbf{f}^\alpha), (\bar{M}^{\alpha+1}, \bar{\mathbf{J}}^{\alpha+1}, \mathbf{f}^{\alpha+1}))$ strictly S_α-obeys $\mathfrak{g}$.
2A) In part (2) we say S-obeys$_1$ when $S_\alpha \subseteq S$ mod $\mathscr{D}_\partial$ for $\alpha \in u \cup \{\delta\}$. Similarly for $\bar{S}'$-obey$_1$ when $\bar{S}' = \langle S'_\alpha : \alpha \in u' \rangle$ and $\alpha \in u \cup \{\delta\} \Rightarrow S^*_\alpha \subseteq S_\alpha$ and $u \cup \{\delta\} \subseteq u'$.
2B) In part (2) we say strictly S-obeys$_1$ when this holds in each case.
3) We say "almost$_1$ every $(\bar{M}, \bar{\mathbf{J}}, \mathbf{f}) \in K^{\rm qt}_{\mathfrak{u}}$ satisfies Pr" <u>when</u> there is a function $\mathfrak{g}$ witnessing it, which means (note: the use of "obey" guarantees $\mathfrak{g}$ is as in part (2) and not as implicitly required on $\mathfrak{h}$ in part (0)):

- (c) if $\langle (\bar{M}^\alpha, \bar{\mathbf{J}}^\alpha, \mathbf{f}^\alpha) : \alpha \le \delta \rangle$ is $\le^{\rm qs}_{\mathfrak{u}}$-increasing continuous obeying $\mathfrak{g}$ and $\delta < \partial^+$ a limit ordinal <u>then</u> $(\bar{M}^\delta, \bar{\mathbf{J}}^\delta, \mathbf{f}^\delta)$ satisfies the property Pr.

3A) We add "above $(\bar{M}, \bar{\mathbf{J}}, \mathbf{f}')$" <u>when</u> we demand in clause (c) that $(\bar{M}^0, \bar{\mathbf{J}}^0, \mathbf{f}^0) = (\bar{M}', \bar{\mathbf{J}}', \mathbf{f}')$.
3B) We replace[16] almost$_1$ by S-almost$_2$ <u>when</u> we require that the sequence "strictly S-obeys $\mathfrak{g}$".
3C) We replace[17] almost$_1$ by almost$_2$ <u>when</u> for every stationary $S \subseteq \partial, S$-almost$_1$ every triple $(\bar{M}, \bar{\mathbf{J}}, \mathbf{f}) \in K^{\rm qt}_{\mathfrak{u}}$ satisfies Pr; and "S-almost$_2$" we ?.

1.23 Definition. 1) For $\mathfrak{h}$ as[18] in Definition 1.22(0) we define $K^{\mathfrak{u},\mathfrak{h}}_{\partial^+}$ as the class of models M such that for some $\le^{\rm qs}_{\mathfrak{u}}$-increasing continuous sequence $\mathbf{x} = \langle (\bar{M}^\alpha, \bar{\mathbf{J}}^\alpha, \mathbf{f}^\alpha) : \alpha < \partial^+ \rangle$ of members of $K^{\rm qt}_{\mathfrak{u}}$ such that a club of $\delta < \partial^+, \mathbf{x} \restriction (\delta+1)$ obeys $\mathfrak{h}$ is the sense of part (0) of Definition 1.22 respectively, we have $M = \cup\{M^\alpha_\partial : \alpha < \partial^+\}$.
2) For $\mathfrak{g}$ as in Definition 1.22(1),(2) we define $K^{\mathfrak{u},\mathfrak{g}}_{\partial^+}$ similarly.
3) We call $\mathfrak{h}$ as in 1.22(0) appropriate$_3$ or 3-appropriate and $\mathfrak{g}$ as in Definition 1.22(1),(2) we call appropriate$_\ell$ or ℓ-appropriate for $\ell = 1, 2$; we may add "$\mathfrak{u}$-"if not clear from the context.

[16]again assume that all elements are ordinals $< \partial^+$

[17]if we replaced it by "for a set of δ's which belongs to $\mathscr{D}$", $\mathscr{D}$ a normal filter on ∂, the difference is minor.

[18]we shall assume that no $\mathfrak{h}$ is both as required in Definition 1.22 and as required in Definition 1.23(0).

4) As in parts (1),(2) for $\mathfrak{h}$ as in (any relevant part of) Definition 1.24 below.
5) Also $K^{\mathfrak{u},\bar{\mathfrak{h}}}_{\partial^+} = \cap\{K^{\mathfrak{u},\mathfrak{h}_\varepsilon}_{\partial^+} : \varepsilon < \ell g(\bar{\mathfrak{h}})\}$ where each $K^{\mathfrak{u},\mathfrak{h}_\varepsilon}_{\partial^+}$ is well defined.

1.24 Definition. 1) We say $\langle(\bar{M}^\zeta, \bar{\mathbf{J}}^\zeta, \mathbf{f}^\zeta) : \alpha < \alpha_*\rangle$ does obey$_0$ (or 0-obey) the function $\mathfrak{h}$ in ζ <u>when</u> $\xi + 1 < \alpha_*$ and (if $\alpha_* = 2$ we can omit ζ):

- (a) $(\bar{M}^\varepsilon, \bar{\mathbf{J}}^\varepsilon, \mathbf{f}^\varepsilon) \in K^{\rm qt}_{\mathfrak{u}}$ is $\le^{\rm qt}_{\mathfrak{u}}$-increasing continuous (with ε)
- (b) M^ζ_∂ and even $M^{\zeta+1}_\partial$ has universe an ordinal $< \partial^+$
- (c) there is a club E of ∂ and sequence $\langle \mathbf{I}_\alpha : \alpha < \partial\rangle$ witnessing $(\bar{M}^\zeta, \bar{\mathbf{J}}^\zeta, \mathbf{f}^\zeta) \le^{\rm at}_{\mathfrak{u}} (\bar{M}^{\zeta+1}, \bar{\mathbf{J}}^{\zeta+1}, \bar{\mathbf{J}}, \mathbf{f}^{\zeta+1})$ such that $(M^\zeta_{\min(E)}, M^{\zeta+1}_{\min(E)}, \mathbf{I}_{\min(E)}) = \mathfrak{h}(\langle(\bar{M}^\xi, \bar{\mathbf{J}}^\xi, \mathbf{f}^\xi) : \xi \le \zeta\rangle) \in \text{ FR}^+_1$.

2) We say that $\mathfrak{h}$ is $\mathfrak{u}$-appropriate$_0$ or $\mathfrak{u} - 0$-appropriate <u>when</u>: $\mathfrak{h}$ has domain and range as required in part (1), particularly clause (c). We may say 0-appropriate or appropriate$_0$ when $\mathfrak{u}$ is clear from the context and we say "$(\bar{M}^\zeta, \dots), (\bar{M}^{\zeta+1}, \dots)$ does 0-obeys $\mathfrak{h}$".
2A) We say the function $\mathfrak{h}$ is $\mathfrak{u}$-1-appropriate <u>when</u> its domain and range are as required in Definition 1.22(3); in this case $S_{\mathfrak{h}} = S$.
2B) We say the function $\mathfrak{h}$ is $\mathfrak{u}$-2-appropriate for S <u>when</u> $S \subseteq \partial$ is stationary and its domain and range are as required in Definition 1.22(3B), i.e. 1.22(3).
2C) If in (2B) we omit S this means that $\bar{\mathfrak{h}} = \langle \mathfrak{h}_S : S \subseteq \partial \text{ is stationary}\rangle$, each $\mathfrak{h}_S$ as above.
3) For 0-appropriate $\mathfrak{h}$ we define $\mathfrak{K}^{\mathfrak{u},\mathfrak{h}}_{\partial^+}$ to be the family of models M, with universe ∂^+ for simplicity, as the set of models of the form $\cup\{M^\zeta_{\partial^+} : \zeta < \partial^+\}$ where $\langle(\bar{M}^\zeta, \bar{\mathbf{J}}^\zeta, \mathbf{f}^\zeta) : \zeta < \partial^+\rangle$ is $\le_{\rm qt}$-increasing continuous and 0-obeys $\mathfrak{h}$ in ζ for unboundedly many $\zeta < \partial^+$. Similarly for the other $\mathfrak{h}$, see below.
4) We say $\mathfrak{h}$ is $\mathfrak{u} - \{0,2\}$-appropriate or $\mathfrak{u}$-appropriate for $\{0,2\}$ if $\mathfrak{h} = \mathfrak{h}_0 \cup \mathfrak{h}_2$ and $\mathfrak{h}_\ell$ is ℓ-appropriate for $\ell = 0,2$; we may omit $\mathfrak{h}$ when clear from the context.
5) For a $\{0,1\}$-appropriate $\mathfrak{h}$ letting $\mathfrak{h}_0, \mathfrak{h}_1$ be as above we say $\langle(\bar{M}^\alpha, \bar{\mathbf{J}}^\alpha, \mathbf{f}^\alpha) : \alpha < \alpha(*)\rangle$ does $\{0,1\}$-obeys $\mathfrak{h}$ in $\zeta < \alpha(*)$ when $((\bar{M}^\zeta, \bar{\mathbf{J}}^\zeta, \mathbf{f}^\zeta), (\bar{M}^{\zeta+1}, \bar{\mathbf{J}}^{\zeta+1}, \mathbf{f}^\zeta))$ does ℓ-obey $\mathfrak{h}_\ell$ for $\ell = 0,2$. We say strictly $\{0,2\} - S$-obeys $\mathfrak{h}$ in ζ when for stationary $S \subseteq \partial$, for unboundedly many $\zeta < \alpha(*)$ the pair 0-obeys $\mathfrak{h}_0$ and strictly $1 - S$-obeys $\mathfrak{h}_1$.
6) For a $\{0,2\}$-appropriate $\mathfrak{h}$, we say $\langle(\bar{M}, \bar{\mathbf{J}}^\alpha, \mathbf{f}^\alpha) : \alpha < \delta \le \partial^+\rangle$ does $\{0,2\}$-obey $\mathfrak{h}$ when this holds for some stationary $S \subseteq \partial$ for unboundedly many $\zeta < \delta$ the sequence strictly $\{0,1\} - S$-obey $\mathfrak{h}$. Similarly we define "the sequence $\{0,2\} - S$-obeys $\mathfrak{h}$".
7) "$\{0,2\}$-almost every $(\bar{M}, \bar{\mathbf{J}}, \mathbf{f})$ (or every $(\bar{M}, \bar{\mathbf{J}}, \mathbf{f})$ above $(\bar{M}^*, \bar{\mathbf{J}}^*, \mathbf{f}^*)$)" is defined similarly to Definition 1.22.

1.25 Observation. 1) For any $\varepsilon^* < \partial^+$ and sequence $\langle \mathfrak{h}_\varepsilon : \varepsilon < \varepsilon^* \rangle$ of 3-appropriate $\mathfrak{h}$, there is an 3-appropriate $\mathfrak{h}$ such that $\mathfrak{K}^{\mathfrak{u},\mathfrak{h}}_{\partial^+} \subseteq \cap\{K^{\mathfrak{u},\mathfrak{h}_\varepsilon}_{\partial^+} : \varepsilon < \varepsilon^*\}$ and similarly for $<^{\rm qs}_{\mathfrak{u}}$-increasing sequences of $K^{\rm qt}_{\mathfrak{u}}$ length $< \partial^+$.
2) $K^{\mathfrak{u},\mathfrak{h}}_{\partial^+} \subseteq K^{\mathfrak{u},*}_{\partial^+}$ for any 3-appropriate function $\mathfrak{h}$.
3) Similarly to parts (1)+(2) for $\mathfrak{g}$ as in Definition 1.22(2).
4) Similarly to parts (1) + (2) for $\{0,2\}$-appropriate $\mathfrak{h}$, see Definition 1.24(4),(5),(6).

1.26 Remark. 1) Concerning 1.25, if in Definition 1.22(1)(c) we do not allow $\odot'$, then we better[19] in 1.22(2) add $\bar S = \langle S^\varepsilon : \varepsilon < \partial\rangle$ such that: $S^\varepsilon \subseteq \partial$ is stationary, $\varepsilon < \zeta < \lambda \Rightarrow S^\varepsilon \cap S^\zeta = \emptyset$ and $\varepsilon < \partial \wedge \alpha \in u \cup \{\delta\} \Rightarrow S^\varepsilon \cap S_\alpha$ is stationary.
2) A priori "almost$_3$" look the most natural, but we shall use as our main case "$\{0,2\}$-almost". We try to explain below.
3) Note that

- (a) in the proof of e.g. 10.10 we use $K^{\rm rt}_{\mathfrak{u}}$ not $K^{\rm qt}_{\mathfrak{u}}$, i.e. carry $\bar{\mathbb{F}}$; this does not allow us the freedom which "almost$_3$" require
- (b) model theoretically here usually there is a special model in $K^{\mathfrak{u}}_\partial$, normally the superlimit or saturated one, and we try to take care building the tree $\langle(\bar M_\eta, \bar{\mathbf{J}}_\eta, \mathbf{f}_\eta, (\bar{\mathbb{F}}_\eta)) : \eta \in {}^{\partial>}(2^\partial)\rangle$ that, e.g. $\eta \in {}^\gamma(2^\partial) \wedge \partial | \gamma \Rightarrow M^\eta_\partial$ is saturated.

In the 'almost$_3$" case this looks straight; in successor of successor cases we can take care.
4) We like to guarantee that for "almost" all $(\bar M^\eta, \bar{\mathbf{J}}^\eta, \mathbf{f}^\eta)$ the model $M^\eta_\partial \in K^{\mathfrak{u},*}_\partial$ is saturated so that we have essentially one case. If we allow in the "almost", for, e.g. $\gamma+2$, to choose some initial segment in $(\bar M^\eta, \bar{\mathbf{J}}^\eta, \mathbf{f}^\eta)$ for η of length $\gamma+1$, this guarantees saturation of M^η_∂ if ${\rm cf}(\ell g(\eta)) = \partial$, but

- (c) set theoretically we do not know that $S^{\partial^+}_\partial = \{\delta < \partial^+ : {\rm cf}(\delta) = \partial\}$ is not in the relevant ideal (in fact, even under GCH, $\diamondsuit_{S^{\partial^+}_\partial}$ may fail)
- (d) if K_∂ is categorical, there is no problem. However, if we know less, e.g. that there is a superlimit one, or approximation, using the almost$_2$, in $\gamma = \gamma'+2$, we can guarantee that M_η for $\eta \in {}^\gamma(2^\partial)$ is up to isomorphism the superlimit one
- (e) we may conclude that it is better to work with $K^{\rm rt}_{\mathfrak{u}}$ rather than $K^{\rm qt}_{\mathfrak{u}}$, see Definition 10.3(1). This is true from the point of view of the construction but it is model theoretically less natural.

[19]in the cases we would like to apply 1.25 there is no additional price for this.

5) We may in Definition 1.24 demand on $\langle (\bar M^\alpha, \bar{\mathbf{J}}^\alpha, \mathbf{f}^\alpha) : \alpha < \delta_* \rangle$ satisfies several $\mathfrak{h}$'s of different kinds say of $\{0,2\}$ and of 3; make little difference.
6) In the usual application here for $\mathfrak{u} = \mathfrak{u}^\ell_{\mathfrak{s}}$ for some $\mathfrak{g}$, if $\langle (M^\alpha, \bar{\mathbf{J}}^\alpha, \mathbf{f}^\alpha) : \alpha \le \delta \rangle$ is $\le^{\rm qt}_{\mathfrak{u}}$-increasing continuous and $\delta = u, u := \{\alpha < \delta : ((\bar M^\alpha, \bar{\mathbf{J}}^\alpha, \mathbf{f}^\alpha), (\bar M^{\alpha+1}, \bar{\mathbf{J}}^{\alpha+1}, \mathbf{f}^{\alpha_1}))$ does strictly S-obey $\mathfrak{g}\}$, <u>then</u> M^δ_∂ is saturated. But without this extra knowledge, the fact that for $\alpha \in u$ we may have S_α disjoint to other may be hurdle. But using "strictly obey$_1$" seems more general and the definition of "almost$_2$" fits this feeling.

§2 Coding properties and non-structure

We now come to the definition of the properties we shall use as sufficient conditions for non-structure starting with Definition 2.2; in this section and §3 we shall define also some relatives needed for sharper results, those properties have parallel cases as in Definition 2.2.

2.1 Hypothesis. We assume $\mathfrak{u}$ be a nice construction framework and, τ a weak $\mathfrak{u}$-sub-vocabulary, see Definition 1.8(1).

Remark. The default value is $\tau_{\mathfrak{u}} = \tau(\mathfrak{K}_{\mathfrak{u}})$ or better the pair $(\tau, \tau_{\mathfrak{u}})$ such that $\tau_{\mathfrak{u}} = \tau'$, as in Definition 1.8(1) and 2.8(1),(2); see also $\mathfrak{u}$ has faked equality, see 3.17 later.

Among the variants of weak τ-coding in Definition 2.2 the one we shall use most is 2.2(5), "$\mathfrak{u}$ has the weak τ-coding$_1$ above $(\bar M^*, \bar{\mathbf{J}}^*, \mathbf{f}^*)$".

2.2 Definition. 1) We say that $M \in K_{\mathfrak{u}}$ has the weak τ-coding$_0$-property (in $\mathfrak{u}$) <u>when</u>:

- (A) if $N, \mathbf{I}$ are such that $(M, N, \mathbf{I}) \in \mathrm{FR}^+_1$ <u>then</u> $(M, N, \mathbf{I})$ has the weak τ-coding$_0$ property,
  where:
- (B) $(M, N, \mathbf{I})$ has the weak τ-coding$_0$ property <u>when</u> we can find $(M_*, N_\ell, \mathbf{I}_\ell) \in \mathrm{FR}_1$ for $\ell = 1, 2$ satisfying
  - (a) $(M, N, \mathbf{I}) \le^1_{\mathfrak{u}} (M_*, N_\ell, \mathbf{I}_\ell)$ for $\ell = 1, 2$
  - (b) $M_* \cap N = M$ (follows)
  - (c) N_1, N_2 are τ-incompatible amalgamations of M_*, N over M in $K_{\mathfrak{u}}$, (see Definition 1.8(4)).

1A) We say that $(M, N, \mathbf{I}) \in \mathrm{FR}^+_1$ has the true weak τ-coding$_0$ <u>when</u>: if $(M, N, \mathbf{I}) \le^1_{\mathfrak{u}} (M', N', \mathbf{I})$ then $(M', N', \mathbf{I}')$ has the weak τ-coding$_0$ property, i.e. satisfies the requirement in clause (B) of part (1).

1B) $\mathfrak{u}$ has the explicit weak τ-coding$_0$ property <u>when</u> every $(M, N, \mathbf{I}) \in \mathrm{FR}^+_{\mathfrak{u}}$ has the weak τ-coding property.
2) $(\bar{M}^*, \bar{\mathbf{J}}^*, \mathbf{f}^*) \in K^{\mathrm{qt}}_{\mathfrak{u}}$ has the weak τ-coding$_0$ property <u>when</u>: for a club of $\delta < \partial$, not only $M = M^*_\delta$ has the true weak τ-coding$_0$-property <u>but</u> in clause (B) of part (1) above we demand $M_* \leq_{\mathfrak{u}} M^*_\gamma$ for any $\gamma < \partial$ large enough.
3) We say that $(\bar{M}, \bar{\mathbf{J}}, \mathbf{f}) \in K^{\mathrm{qt}}_{\mathfrak{u}}$ has the weak τ-coding$_1$ property[20] <u>when</u>: (we may omit the superscript 1): recalling $M_\partial = \cup\{M_\alpha : \alpha < \partial\}$, there are $\alpha(0) < \partial$ and $N_0, \mathbf{I}_0$ such that $(M_{\alpha(0)}, N_0, \mathbf{I}_0) \in \mathrm{FR}_1, N_0 \cap M_\partial = M_{\alpha(0)}$ and for a club of $\alpha(1) < \partial$, if $(M_{\alpha(0)}, N_0, \mathbf{I}_1) \leq^1_{\mathfrak{u}} (M_{\alpha(1)}, N_1, \mathbf{I}_1)$ satisfies $N_1 \cap M_\partial = M_{\alpha(1)}$ <u>then</u> there are $\alpha(2) \in (\alpha(1), \partial)$ and $N^\ell_2, \mathbf{I}^\ell_2$ for $\ell = 1, 2$ such that $(M_{\alpha(1)}, N_1, \mathbf{I}_1) \leq^1_{\mathfrak{u}} (M_{\alpha(2)}, N^\ell_2, \mathbf{I}^\ell_2)$ for $\ell = 1, 2$ and N^1_2, N^2_2 are τ-incompatible amalgamations of $M_{\alpha(2)}, N_1$ over $M_{\alpha(1)}$ recalling Definition 1.8(4).
4) We say that $(\bar{M}, \bar{\mathbf{J}}, \mathbf{f})$ has the S-weak τ-coding$_1$ property <u>when</u>: S is a stationary subset of ∂ and for some club E of ∂ the demand in (3) holds restricting ourselves to $\alpha(1) \in S \cap E$.
5) We say that $\mathfrak{u}$ has the weak τ-coding$_k$ property <u>when</u>: $\{0, 2\}$-almost every $(\bar{M}, \bar{\mathbf{J}}, \mathbf{f}) \in K^{\mathrm{qt}}_{u}$ has the weak τ-coding$_k$ property; omitting k means $k = 1$. Similarly for "above $(\bar{M}, \bar{\mathbf{J}}, \mathbf{f}) \in K^{\mathrm{qt}}_{\mathfrak{u}}$". Similarly for "$S$-weak".

The following theorem uses a weak model theoretic assumption, but the price is a very weak but still undesirable, additional set theoretic assumption (i.e. clause (c)), recall that $\mu_{\mathrm{unif}}(\partial^+, 2^\partial)$ is defined in 0.4(7), see 9.4.

2.3 Theorem. *We have $\dot{I}_\tau(\partial^+, K^{\mathfrak{u}}_{\partial^+}) \geq \mu_{\mathrm{unif}}(\partial^+, 2^\partial)$, moreover for any $\mathfrak{u}$-0-appropriate $\mathfrak{h}$ (see Definition 1.24) and even $\{0, 2\}$-appropriate $\mathfrak{h}$ (see Definition 1.24(3),(7) and Definition 1.23) we have $\dot{I}(K^{\mathfrak{u},\mathfrak{h}}_{\partial^+}) \geq \mu_{\mathrm{unif}}(\partial^+, 2^\partial)$, <u>when</u>:*

(a) $2^\theta = 2^{<\partial} < 2^\partial$

(b) $2^\partial < 2^{\partial^+}$

(c) *the ideal* $\mathrm{WDmId}(\partial)$ *is not ∂^+-saturated*

(d) *$\mathfrak{u}$ has the weak τ-coding (or just the S-weak τ-coding property above some triple $(\bar{M}, \bar{\mathbf{J}}, \mathbf{f}) \in K^{\mathrm{qt}}_{\mathfrak{u}}$ with $\mathrm{WDmId}(\partial) \restriction S$ not ∂^+-saturated and $S \subseteq \mathbf{f}^{-1}\{0\}$).*

Proof. This is proved in 10.10. $\square_{2.3}$

[20]the difference between coding$_0$ and coding$_1$ may seem negligible but it is crucial, e.g. in 4.1

2.4 Remark. 1) Theorem 2.3 is used in 4.1, 4.3(1), 4.20, 4.16 and 4.28, for 4.3(2) we use the variant 2.5. We could use Theorem 2.7 below to get a somewhat stronger result.
In other words, e.g. it is used for "the minimal types are not dense in $\mathscr{S}(M)$ for $M \in \mathfrak{K}_\lambda$" for suitable $\mathfrak{K}$, see 4.1 (and Chapter VI or the older [Sh 576], [Sh 603]).
2) We may think that here at a minor set theoretic price (clause (c)), we get the strongest model theoretic version.
3) We can in 2.3 replace $\mathfrak{h}$ by $\bar{\mathfrak{h}}$, a sequence of $\{0,2\}$-appropriate $\mathfrak{h}$'s of length $\le \partial^+$.
4) In part (3), we can fix a stationary $S \subseteq \partial$ such that $\mathrm{WDmId}(\partial) + S$ is not ∂^+-saturated (so ∂ is not in it) and restrict ourselves to strict S-obeying.
5) We can replace assumption (d) of 2.3 by

- $(d)'$ for some $\mathscr{D}$
  - (i) $\mathscr{D}$ is a normal filter on ∂ disjoint to $\mathrm{WDmId}(\partial)$; moreover $(\forall A \in \mathscr{D})(\exists B)[B \subseteq A \wedge \partial \backslash B \in D \wedge B \in (\mathrm{WDmId}(\partial))^+]$
  - (ii) almost$_2$ every $(\bar{M}, \bar{\mathbf{J}}, \mathbf{f}) \in K^{\mathrm{qt}}_{\mathscr{D}}$ has the weak τ-coding property (even just above some member of $K^{\mathrm{qt}}_{\mathscr{D}}$).

A variant is

2.5 Claim. *In Theorem 2.3 we can weaken the assumption to "$\mathfrak{u}$ has a weak τ-coding$_2$", see below.*

Proof. As in 10.10. $\square_{2.5}$

2.6 Definition. 1) We say that $\mathfrak{u}$ has the S-weak τ-coding$_2$ property (or the S-weak game τ-coding property) [above $(\bar{M}^*, \mathbf{J}^*, \mathbf{f}^*) \in K^{\mathrm{qt}}_{\mathfrak{u}}$] <u>when</u> $\{0,2\}$-almost every $(\bar{M}, \bar{\mathbf{J}}, \mathbf{f}) \in K^{\mathrm{qt}}_{\mathfrak{s}}$ [above $(\bar{M}^*, \bar{\mathbf{J}}^*, \mathbf{f}^*)$] has it.
2) We say $(\bar{M}, \bar{\mathbf{J}}, \mathbf{f}) \in K^{\mathrm{qt}}_{\mathfrak{u}}$ has the S-weak game τ-coding$_2$ property <u>or</u> S-weak τ-coding$_2$ property for a stationary set $S \subseteq \partial$ (omitting S means for every such S) <u>when</u>, recalling $M_\partial = \cup\{M_\alpha : \alpha < \partial\}$, in the following game $\eth_{\mathfrak{u},S}(\bar{M}, \bar{\mathbf{J}}, \mathbf{f})$, the Coder player has a winning strategy where:

- $(*)_1$ a play of $\eth_{\mathfrak{u},S}$ last ∂ moves after the ε-th move a tuple $(\alpha_\varepsilon, e_\varepsilon, \bar{N}^\varepsilon, \bar{\mathbf{J}}^\varepsilon, \mathbf{f}^\varepsilon, \bar{I}^\varepsilon)$ is chosen such that:
  - (a) $\alpha_\varepsilon < \partial$ is increasing continuous
  - (b) e_ε is a closed subset of α_ε such that $\zeta < \varepsilon \Rightarrow \alpha_\zeta \in e \wedge e_\zeta = e_\varepsilon \cap \alpha_\zeta$

(c) $\mathbf{f}^\varepsilon$ is a function with domain e_ε such that $\alpha+\mathbf{f}^\varepsilon(\alpha) < \min(e_\varepsilon \cup \{\alpha_\varepsilon\} \backslash \alpha)$ and $\mathbf{f}^\varepsilon(\alpha) \geq \mathbf{f}(\alpha)$

(d) $u_\varepsilon = \cup\{[\alpha, \alpha+\mathbf{f}^\varepsilon(\alpha)] : \alpha \in e_\varepsilon\} \cup \{\alpha_\varepsilon\}$ and $u^-_\varepsilon = \cup\{[\alpha, \alpha+\mathbf{f}^\varepsilon(\alpha))) : \alpha \in e_\varepsilon\}$

(e) $\bar{N}^\varepsilon = \langle N_\alpha : \alpha \in u_\varepsilon \rangle$ and $\mathbf{J}^\varepsilon = \langle \mathbf{J}_\alpha : \alpha \in u^-_\varepsilon \rangle$ and $\bar{\mathbf{I}}^\varepsilon = \langle \mathbf{I}_\alpha : \alpha \in u_\varepsilon \rangle$

(f) $\langle (M_\alpha, N_\alpha, \mathbf{I}_\alpha) : \alpha \in u_\varepsilon \rangle$ is $\leq^1_{\mathfrak{u}}$-increasing

(g) $N^\varepsilon_\alpha \cap M_\partial = M_\alpha$ for $\alpha \in u_\varepsilon$

(h) $\bar{\mathbf{J}}^\varepsilon = \langle \mathbf{J}^*_\alpha : \alpha \in u^-_\varepsilon \rangle$

(i) $(M_\alpha, M_{\alpha+1}, \mathbf{J}_\alpha) \leq^2_{\mathfrak{u}} (N_\alpha, N_{\alpha+1}, \mathbf{J}^*_\alpha)$ for $\alpha \in u^-_\varepsilon$

(j) the coder chooses $(\alpha_\varepsilon, e_\varepsilon, \bar{N}^\varepsilon, \bar{\mathbf{J}}^\varepsilon, \bar{\mathbf{I}}^\varepsilon, \mathbf{f}^\varepsilon)$ if $\varepsilon = 0$ or $\varepsilon = \zeta+1, \zeta$ a limit ordinal $\notin S$, and otherwise the anti-coder chooses

$(*)_2$ in the end the Coder wins the play <u>when</u> for a club of $\varepsilon < \partial$, if $\varepsilon \in S$, then the triple $(M_{\alpha_\varepsilon}, N_\varepsilon, \mathbf{I}_\varepsilon)$ has the weak τ-coding$_0$ property, i.e. satisfies clause (B) of Definition 2.2(1), moreover such that $M \leq_{\mathfrak{K}} M_\partial$.

We can also get "no universal" over $M_\partial \in \mathfrak{K}^{\mathfrak{u}}_\partial$ (suitable for applying 9.2).

2.7 Claim. *If* $(\bar{M}, \mathbf{J}, \mathbf{f}) \in K^{\rm qt}_{\mathfrak{u}}, M = \cup\{M_\alpha : \alpha < \partial\}$ *and* $M \leq_{\mathfrak{u}} N_\varepsilon \in \mathfrak{K}_{\leq \mu}$ *for* $\varepsilon < \varepsilon^* < \mu^+$ <u>*then*</u> *there is* $(\bar{M}', \bar{\mathbf{J}}', \mathbf{f})$ *satisfying* $(\bar{M}, \bar{\mathbf{J}}, \mathbf{f}) \leq^{\mathfrak{u}}_{\rm at} (\bar{M}', \bar{\mathbf{J}}', \mathbf{f}')$ *such that* $M'_\partial = \cup\{M_\alpha : \alpha < \partial\} \in K_\partial$ *cannot be* $\leq_{\mathfrak{K}[\mathfrak{u}]}$*-embedded into* N_ε *for* $\varepsilon < \partial$ *over* M_∂ <u>*provided that*</u>*:*

(a),(b),(d) *as in 2.3*

(e) ${}^\partial 2$ *is not the union of* $\mathrm{cov}(\mu, \partial^+, \partial^+, 2)$ *sets from* $\mathrm{WDmTId}(\partial, 2^{<\partial})$.

Proof. As in the proof of 10.10, anyhow not used. $\square_{2.7}$

<u>2.8 Exercise</u>: 1) [Definition] Call $\mathfrak{u}$ a semi-nice construction framework <u>when</u> in Definition 1.2 we omit clause $(D)_\ell(d)$ and the disjointness demands $(E)_\ell(b)(\beta)$
2) For $\mathfrak{u}$ as above we define $\mathfrak{u}'$ as follows:

(a) $\mathfrak{K}_{\mathfrak{u}'}$ is as in Definition 1.10(1)

(b) $\mathrm{FR}^\ell_{\mathfrak{u}'} = \{(M, N, \mathbf{J}) : M \leq_{\mathfrak{K}'_{\mathfrak{u}}} N$ so both of cardinality $< \partial$ and $\mathbf{J} \subseteq N \backslash M$ and letting $M^* = M/=_\tau, N^* = N/=_\tau$ and $\mathbf{J}^* = \{c/=^N_\tau : c \in \mathbf{J}$ and $(c/=_\tau) \notin M/=_\tau\}$ we have $(M^*, N^*, \mathbf{J}^*) \in \mathrm{FR}^\ell_{\mathfrak{u}}\}$
pedantically $=_\tau$ means $=^N_\tau$ (even $M/=^N_\tau$)

(c) $(M_1, N_1, \mathbf{J}_1) \le^\ell_{\mathfrak{u}'} (M_2, N_2, \mathbf{J}_1)$ iff $M_1 \le_{\mathfrak{u}'} M_2 \le_{\mathfrak{u}'} N_2, M_1 \le_{\mathfrak{u}'} N_1 \le_{\mathfrak{u}'} N_2, \mathbf{J}_1 \subseteq \mathbf{J}_2, M_2 \cap N_1 = M_1$ and $(M^*_1, N^*_1, \mathbf{J}_1) \le^\ell_{\mathfrak{u}} (M^*_2, N^*_2, \mathbf{J}^*)$ when we define them as in clause (b).

3) (Claim) If $\mathfrak{u}$ is a semi-nice construction framework then $\mathfrak{u}'$ is a nice construction framework.
4) [Definition] For $\mathfrak{u}$ a semi-nice construction framework we define $\mathfrak{u}''$ as in part (2) except that in clause (b) we demand $c \in \mathbf{J} \wedge a \in M \Rightarrow N \models \neg(a =_\tau c)$.
5) [Claim] If $\mathfrak{u}$ is a nice, [semi-nice], [semi-nice satisfying (D)(d)] construction framework then $\mathfrak{u}''$ is a nice, [semi-nice], [nice] construction framework.

Discussion: We now phrase further properties which are enough for the desired conclusions under weaker set theoretic conditions. The main case is vertical coding (part (4) but it relies on part (1) in Definition 2.9). On additional such properties, see later.

In the "vertical coding" version (see Definition 2.9 below), we strengthen the "density of τ-incompatibility" such that during the proof we do not need to preserve "$\mathbf{f}^{-1}_\eta\{0\}$ is large" even allowing $\mathbf{f}^{-1}_\eta\{0\} = \emptyset$.

We may say that "vertically" means that given $(\bar M^1, \bar{\mathbf{J}}^1, \mathbf{f}^1) \in K^{\rm qt}_{\mathfrak{u}}$ building $M^2_\alpha, \mathbf{J}^2_\alpha, \mathbf{I}^2_\alpha$ by induction on $\alpha < \partial$, arriving to some limit δ, we are committed to $M^2_{\alpha+i}$ for $i \le \mathbf{f}^1(\delta)$, but still like to have freedom in determining the type of M^2_δ over $\cup\{M^1_\beta : \beta < \partial\}$ (see more in the proof of Theorem 8.6 and Definition 8.7 on delayed uniqueness, which express failure of this freedom). In other words the property we have is a delayed version of the weak coding.
As usual, always $\mathfrak{u}$ is a nice construction framework.

2.9 Definition. 1) We say that $(\bar M^1, \bar{\mathbf{J}}^1) = (\langle M^1_i : i \le \beta\rangle, \langle \mathbf{J}^1_i : i < \beta\rangle)$ has the vertical τ-coding$_0$ property (in $\mathfrak{u}$) when:

$(A)(a)$ $\beta < \partial$

$(b)_1$ M^1_i is $\le_{\mathfrak{u}}$-increasing continuous for $i \le \beta$

$(c)_1$ $(M^1_i, M^1_{i+1}, \mathbf{J}^1_i) \in$ FR$_2$ for $i < \beta$

(B) if $(\langle M^2_i : i \le \beta\rangle, \langle \mathbf{J}^2_i : i < \beta\rangle, \langle \mathbf{I}_i : i \le \beta\rangle)$ satisfies $\circledast_1$ below, then we can find $\gamma_\ell, M^1_*, \mathbf{I}^\ell_*$ and $M^{2,\ell}_i$ (for $i \in (\beta, \gamma_\ell]$) and $\mathbf{J}^{2,\ell}_i$ (for $i \in [\beta, \gamma_\ell)$), for $\ell = 1, 2$ satisfying $\circledast_2$ below where, letting $M^{2,\ell}_i = M^2_i$ for $i \le \beta$ and $\mathbf{J}^{2,\ell}_i = \mathbf{J}^2_i$ for $i < \beta$, we have:

$\circledast_1$ (d) $M^2_i\,(i \le \beta)$ is $\le_{\mathfrak{u}}$-increasing continuous

(e) $M^2_i \cap M^1_\beta = M^1_i$

(f) $\langle (M^1_i, M^2_i, \mathbf{I}_i) : i \le \beta\rangle$ is $\le^1_{\mathfrak{u}}$-increasing continuous and

$(M^1_i, M^2_i, \mathbf{I}_i) \in \mathrm{FR}^+_1$

(g) $(M^1_i, M^1_{i+1}, \mathbf{J}^1_i) \le^2_{\mathfrak{u}} (M^2_i, M^2_{i+1}, \mathbf{J}^2_i) \in \mathrm{FR}_2$ for $i < \beta$

(h) $(M^1_0, M^2_0, \mathbf{I}_0) = (M, N, \mathbf{I})$

$\circledast_2$ $M^{2,1}_{\gamma_1}, M^{2,2}_{\gamma_2}$ are τ-incompatible amalgamation of M^1_*, M^2_0 over M^1_0 in $\mathfrak{K}_{<\partial}$ and for $\ell = 1, 2$ we have

$\circledast_{2,\ell}$ $(a)'$ $\beta < \gamma_\ell < \partial$

$(b)'_1$ $M^1_\beta \le_{\mathfrak{u}} M^1_*$

$(c)'_1$ $(M^{2,\ell}_i, M^{2,\ell}_{i+1}, \mathbf{J}^{2,\ell}_i) \in \mathrm{FR}_2$ for $i < \gamma_\ell$

$(d)'$ $M^{2,\ell}_i$ (for $i \le \gamma_\ell$) is $\le_{\mathfrak{u}}$-increasing continuous

$(e)'$ $M^1_* \le_{\mathfrak{u}} M^{2,\ell}_{\gamma_\ell}$

$(f)'$ $(M^1_\beta, M^2_\beta, \mathbf{I}_\beta) \le^1_{\mathfrak{u}} (M^1_*, M^{2,\ell}_{\gamma_\ell}, \mathbf{I}^\ell_*)$.

1A) We say that $(M, N, \mathbf{I})$ has the vertical τ-coding$_0$ property when: <u>if</u> $(\bar{M}^1, \bar{\mathbf{J}}^1) = (\langle M^1_i : i \le \beta \rangle, \langle \mathbf{J}^1_i : i < \beta \rangle)$ satisfies clause (A) of part (1) and $(\langle M^2_i : i \le \beta \rangle, \langle \mathbf{J}^2_i : i < \beta \rangle, \langle \mathbf{I}_i : i \le \beta \rangle)$ satisfies $\circledast_1$ of clause (B) of part (1) and $(M^1_0, M^2_0, \mathbf{I}_0) = (M, N, \mathbf{I})$ <u>then</u> we can find objects satisfying $\circledast_2$ of clause (B) of part (1).
1B) We say that $(M, N, \mathbf{I})$ has the true vertical τ-coding$_0$ property <u>when</u> it belongs to FR^+_1 and every $(M', N', \mathbf{J}')$ satisfying $(M, N, \mathbf{I}) \le^1_{\mathfrak{u}} (M', N', \mathbf{I}')$ has the vertical τ-coding$_0$ property.
1C) We say that $\mathfrak{u}$ has the explicit vertical τ-coding$_0$ property <u>when</u> for every M for some $N, \mathbf{I}$ the triple $(M, N, \mathbf{I}) \in \mathrm{FR}^+_1$ has the true vertical τ-coding$_0$ property.
2) $(\bar{M}^*, \bar{\mathbf{J}}^*, \mathbf{f}^*) \in K^{\mathrm{qt}}_{\mathfrak{u}}$ has the vertical τ-coding$_0$ property <u>when</u> for a club of $\delta < \partial$, the pair $(\bar{M}^1, \bar{\mathbf{J}}^1) = (\langle M^*_{\delta+i} : i \le \mathbf{f}^*(\delta) \rangle, \langle \mathbf{J}^*_{\delta+1} : i < \mathbf{f}^*(\delta) \rangle)$ satisfies part (1) even demanding $M^1_* \le_{\mathfrak{K}} M^*_\partial$.
3) We say that $(\bar{M}, \bar{\mathbf{J}}, \mathbf{f}) \in K^{\mathrm{qt}}_{\mathfrak{u}}$ has the vertical τ-coding$_1$ property <u>when</u> (we may omit the subscript 1) we can find $\alpha(0) < \partial$ and $(M_{\alpha(0)}, N_*, \mathbf{I}_*) \in \mathrm{FR}_1$ satisfying $N_* \cap M_\partial = M_{\alpha(0)}$ such that: for a club of $\delta < \partial$ the pair $(\bar{M}^1, \bar{\mathbf{J}}^1) = (\langle M_{\delta+i} : i \le \mathbf{f}(\delta) \rangle, \langle \mathbf{J}_{\delta+i} : i < \mathbf{f}(\delta) \rangle)$ satisfies part (1) when in clause (B) where we

(i) restrict ourselves to the case $(M_{\alpha(0)}, N_*, \mathbf{I}_*) \le^1_{\mathfrak{u}} (M^1_0, M^2_0, \mathbf{I}_0)$

(ii) demand that $M^1_* <_{\mathfrak{K}[\mathfrak{u}]} M_\partial$.

4) We say that $(\bar{M}, \bar{\mathbf{J}}, \mathbf{f}) \in K^{\mathrm{qt}}_{\mathfrak{u}}$ has the S-vertical τ-coding$_1$ property <u>when</u>: S is a stationary subset of ∂ and for club E of ∂ the requirement in part (3) holds when we restrict ourselves to $\delta \in S \cap E$.
4A) We say that $(\bar{M}, \bar{\mathbf{J}}, \mathbf{f}) \in K^{\mathrm{qt}}_{\mathfrak{u}}$ has the S-vertical τ-coding$_2$ property as in Definition 2.6.

5) For $k = 0, 1, 2$ we say $\mathfrak{u}$ has the vertical τ-coding$_k$ property <u>when</u> $\{0, 2\}$-almost every $(\bar{M}, \bar{\mathbf{J}}, \mathbf{f}) \in K^{\rm qt}_{\mathfrak{u}}$ has it. If $k = 1$ we may omit it. Similarly adding "above $(\bar{M}^*, \bar{\mathbf{J}}^*, \mathbf{f}^*)$" and/or S-vertical, for stationary $S \subseteq \partial$.

The following observation is easy but very useful.

2.10 Observation. 1) Assume that some $(M, N, \mathbf{I}) \in$ FR^+_1 has the true vertical τ-coding$_0$ property (from Definition 2.9(1B)). If $(\bar{M}, \bar{\mathbf{J}}, \mathbf{f}) \in K^{\rm qt}_{\mathfrak{u}}$ and $M_\partial := \cup\{M_\alpha : \alpha < \partial\}$ is saturated (above λ, for $\mathfrak{K} = \mathfrak{K}^{\mathfrak{u}}$) <u>then</u> $(\bar{M}, \bar{\mathbf{J}}, \mathbf{f})$ has the vertical τ-coding property. See 2.9(2),2.9(2A) used in 4.14(5).
2) If $(\bar{M}, \bar{\mathbf{J}}, \mathbf{f})$ has the vertical τ-coding$_0$ property <u>then</u> it has the vertical τ-coding$_1$ property.
3) Similarly (to part (2)) for weak τ-coding.
4) Recalling $\mathfrak{K}_{\mathfrak{u}}$ has amalgamation (by claim 1.3(1))

- (a) if $M \in \mathfrak{K}_{\mathfrak{u}} \Rightarrow |\mathscr{S}_{\mathfrak{K}}(M)| \leq \partial$ <u>then</u> there is a saturated $M \in K^{\mathfrak{u}}_\partial$
- (b) if every $M \in K^{\mathfrak{u},*}_\partial$ is saturated and every $(M, N, \mathbf{I}) \in$ FR^+_1 has weak τ-coding$_0$, <u>then</u> every $(\bar{M}, \bar{\mathbf{J}}, \mathbf{f}) \in K^{\rm qt}_{\mathfrak{u}}$ has weak τ-coding
- (c) similarly for vertical τ-coding
- (d) similarly replacing "every $M \in K^{\mathfrak{u},*}_\partial$" by "$M_\partial$ is saturated for $\{0, 2\}$-almost every $(\bar{M}, \bar{\mathbf{J}}, \mathbf{f}) \in K^{\rm qt}_{\mathfrak{u}}$" [or just above $(\bar{M}^*, \bar{\mathbf{J}}^*, \mathbf{f}^*) \in K^{\rm qt}_{\mathfrak{u}}$.]

Proof. Should be clear. $\square_{2.8}$

2.11 Theorem. *We have $\dot{I}_\tau(\partial^+, K^{\mathfrak{u}}_{\partial^+}) \geq \mu_{\rm unif}(\partial^+, 2^\partial)$; moreover $\dot{I}(K^{\mathfrak{u},\mathfrak{h}}_{\partial^+}) \geq \mu_{\rm unif}(\partial^+, 2^\partial)$ for any $\{0, 2\}$-appropriate $\mathfrak{h}$ (see Definitions 1.24(2),(7), 1.23) <u>when</u>:*

- (a) $2^\theta = 2^{<\partial} < 2^\partial$
- (b) $2^\partial < 2^{\partial^+}$
- (c) *$\mathfrak{u}$ has the vertical τ-coding$_1$ property (or at least $\mathfrak{u}$ has the S-vertical τ-coding$_\tau$ property above some triple from $K^{\rm qt}_{\mathfrak{u}}$ for stationary $S \in (\mathrm{WDmId}(\partial))^+$ (recall τ is a weak $\mathfrak{u}$-sub-vocabulary, of course, by 2.1)).*

Remark. Theorem 2.11 is used in 4.10 and in 8.6.

Proof. Proved in 10.12. $\square_{2.11}$

* * *

2.12 Discussion: 1) In a sense the following property "horizontal τ-coding" is dual to the previous one "vertical τ-coding", it is "horizontal", i.e. in the ∂^+-direction. This will result in building $(\bar M^\eta, \bar{\mathbf{J}}^n, \bar{\mathbf{f}}^\eta)$ for $\eta \in {}^{\partial^+ >}(2^{<\partial})$ such that letting $M^\eta_\partial = \cup\{M^{\eta \restriction \alpha}_\partial : \alpha < \partial^+\}$ we have $\eta \neq \nu \in {}^{\partial^+}(2^\partial) \Rightarrow M^\eta_\partial, M^\nu_\partial$ are not isomorphic over $M^{<>}_\partial$, so the set theory is simpler.
2) Note that in 2.13(4) below we could ask less than "for a club", e.g. having a winning strategy is the natural game; similarly in other definitions of coding properties, as in Exercise 2.5.

2.13 Definition. 1) We say that $(M_0, M_1, \mathbf{J}_2) \in \mathrm{FR}_2$ has the horizontal τ-coding$_0$ property when: if $\circledast_1$ holds then we can find $N^{+,\ell}_1, \mathbf{I}^\ell_5, \mathbf{J}^\ell_5$ for $\ell = 1, 2$ such that $\circledast_2$ holds when:

$\circledast_1$ (a) $M_0 \le_{\mathfrak{u}} N_0 \le_{\mathfrak{u}} N_1, M_0 \le_{\mathfrak{u}} M_1 \le_{\mathfrak{u}} N_1$

(b) $(M_0, M_1, \mathbf{J}_2) \le^2_{\mathfrak{u}} (N_0, N_1, \mathbf{J}_3)$ so both are from FR_2

(c) $(N_0, N^+_0, \mathbf{I}_4) \in \mathrm{FR}^+_1$ and $N^+ \cap N_1 = N_0$

$\circledast_2$ (α) $(N_0, N^+_0, \mathbf{I}_4) \le^1_{\mathfrak{u}} (N_1, N^{+,\ell}_1, \mathbf{I}^\ell_5)$

(β) $(N_0, N_1, \mathbf{J}_3) \le^2_{\mathfrak{u}} (N^+_0, N^{+,\ell}_1, \mathbf{J}^\ell_5)$

(γ) $N^{+,1}_1, N^{+,2}_1$ are τ-incompatible amalgamations of M_1, N^+_0 over M_0 in $K_{\mathfrak{u}}$.

2) We say that $(\bar M, \bar{\mathbf{J}}, \mathbf{f}) \in K^{\mathrm{qt}}_{\mathfrak{u}}$ has the S-horizontal τ-coding$_0$ property when S is a stationary subset of ∂ and for a club of $\delta \in S$, the triple $(M_\delta, M_{\delta+1}, \mathbf{J}_\delta)$ has it and $\mathbf{f}(\delta) > 0$. If $S = \partial$ we may omit it.
3) We say that $(\bar M, \bar{\mathbf{J}}, \mathbf{f}) \in K^{\mathrm{qt}}_{\mathfrak{u}}$ has the horizontal τ-coding$_1$ property when (we may omit the 1):

$\circledast$ for $\{0, 2\}$-almost every $(\bar M', \bar{\mathbf{J}}', \mathbf{f}') \in K^{\mathrm{qt}}_{\mathfrak{u}}$ satisfying $(\bar M, \bar{\mathbf{J}}, \mathbf{f}) \le^{\mathrm{qs}}_{\mathfrak{u}} (\bar M', \bar{\mathbf{J}}', \mathbf{f}')$ we can find $\alpha < \partial$ and $N, \mathbf{I}^*$ satisfying $(M'_\alpha, N, \mathbf{I}^*) \in \mathrm{FR}^+_2$ and $N \cap M'_\partial = M'_\alpha$ such that

$\boxdot$ for a club of $\delta < \partial$ if $(M'_\alpha, N, \mathbf{I}^*) \le_1 (M'_\delta, N', \mathbf{I}') \in \mathrm{FR}^+_1$ and $N' \cap M'_\partial = M'_\alpha$, then the conclusion in 2.13(1) above holds with $M_{\alpha,i(*)}, M'_{i(*)}, M_\delta, M'_\delta, N', \mathbf{I}', \mathbf{J}_\delta, \mathbf{J}'_\delta$ here standing for $M_0, N_0, M_1, N_1, N^+_0, \mathbf{I}_4, \mathbf{J}_2, \mathbf{J}_3$ there.

4) We replace coding$_1$ by coding$_2$ when in $\boxdot$ we use the game version, as in 2.6.
5) We say $\mathfrak{u}$ has the horizontal τ-coding$_k$ property <u>when</u> some $(\bar{M}, \bar{\mathbf{J}}, \mathbf{f}) \in K^{\rm qt}_{\mathfrak{u}}$ has it.

2.14 Claim. *The coding$_0$ implies the coding$_1$ versions in Definition 2.13 for $(\bar{M}, \bar{\mathbf{J}}, \mathbf{f}) \in K^{\rm qt}_{\mathfrak{u}}$ in Definition 2.13(4) and for $\mathfrak{u}$ in Definition 2.13(5).*

Proof. Should be clear. $\square_{2.14}$

2.15 Theorem. *We have $\dot{I}_\tau(\partial^+, K^{\mathfrak{u}}_{\partial^+}) \ge 2^{\partial^+}$; moreover $\dot{I}(K^{\mathfrak{u},\mathfrak{h}}_{\partial^+}) \ge 2^{\partial^+}$ for any $\{0,2\}$-appropriate $\mathfrak{h}$ (see Definition 1.24(4),(5),(6)), <u>when</u>:*

- (a) $2^\theta = 2^{<\partial} < 2^\partial$
- (b) $2^\partial < 2^{\partial^+}$
- (c) *the ideal* $\mathrm{WDmId}(\partial)$ *is not ∂^+-saturated*
- (d) $\mathfrak{u}$ *has the horizontal τ-coding property (or just the S-horizontal τ-coding$_2$ property for some stationary $S \subseteq \partial$).*

2.16 Remark. 1) Actually not used here.
2) What does this add compared to 2.3? getting $\ge 2^{\partial^+}$ rather than $\ge \mu_{\rm unif}(\partial^+, 2^\partial)$.

Proof. Proved in 10.13. $\square_{2.15}$

§3 Invariant codings

The major notion of this section is (variants of) uq-invariant coding properties. In our context, the point of coding properties is in esssence that their failure gives that there are many of uniqueness triples, $(M, N, \mathbf{J})$ ones, i.e. such that: if $(M, N, \mathbf{I}) \le_1 (M', N'_\ell, \mathbf{I}'')$ for $\ell = 1, 2$ then N'_1, N'_2 are compatible over $N \cup M'$. For uq-invariant we ask for less: if $(M, N, \mathbf{I}) \le (M', N', \mathbf{I}')$ and $\mathbf{d}$ is a $\mathfrak{u}$-free $(\xi, 0)$-rectanle with $(M^{\mathbf{d}}_{0,0}, M^{\mathbf{d}}_{\alpha(\mathbf{d}),0}) = (M, M')$, <u>then</u> we can "lift" $\mathbf{d}''$, i.e. find a $\mathfrak{u}$-free $(\xi + 1, 1)$-rectangle $\mathbf{d}^+$ such that $\mathbf{d}^+ \restriction (\xi, 0) = \mathbf{d}, (M^{\mathbf{d}}_{0,0}, M^{\mathbf{d}}_{0,1}, \mathbf{I}^{\mathbf{d}}_{0,0}) = (M, N, \mathbf{I})$ and $N' \le_{\mathfrak{u}} M^{\mathbf{d}}_{\xi+1,1}$.

So we look at the simplest version, the weak ξ-uq-invariant coding, Definition 3.2, we can consider a "candidate" $(M, N, \mathbf{I}) \in \mathrm{FR}^1_{\mathfrak{u}}$ and challenge $\mathbf{d}$ (so $M^{\mathbf{d}}_{0,0} = M$) and looks for a pair of amalgamation which are incompatile in a specific way, <u>but</u>

unlike in §2, they are not symmetric. One is really not an amalgamation but a family of those exhibiting $\mathbf{d}$ is "liftable", and "promise to continue to do so in the future", in the ∂^+-direction. The real one just has to contradict it.

Another feature is that instead of considering isomorphisms over $M^{\mathbf{d}}_{\alpha(\mathbf{d}),0} \cup N$ we consider isomorphisms over $M^{\mathbf{d}}_{\alpha(\mathbf{d}),0}$ with some remnants of preserving N; more specifically we consider two $\mathfrak{u}$-free rectangles $\mathfrak{d}_1, \mathfrak{d}_2$ which continues the construction in those two ways and demands $M^{\mathbf{d}_1}_{\alpha(\mathbf{d}_1)}$ is mapped onto $M^{\mathbf{d}_2}_{0,\alpha(\mathbf{d}_2)}$.

There are more complicating factors: we have for a candidate $(M, N, \mathbf{I})$, for every $M', M \leq_{\mathfrak{u}} M'$ to find a $\mathfrak{u}$-free $(\xi, 0)$-rectangle $\mathbf{d}$ with $M' = M^{\mathbf{d}}_{0,0}$ such that it will serve against $(M', N', \mathbf{I}')$ whenever $(M, N, \mathbf{I}) \leq_1 (M', N', \mathbf{I}')$, rather than choosing $\mathbf{d}$ after $(N', \mathbf{I}')$ is chosen, i.e. this stronger version is needed. The case $\xi < \partial$ should be clear but still we allow $\xi = \partial$, however then given $(N', \mathbf{I}')$ we take $\mathbf{d} \restriction (\xi', 0)$ for some $\xi' < \xi'$.

We can use only Definition 3.2, Claim 3.3, and Conclusion 3.5, for which "$2^\theta = 2^{<\partial} < 2^\partial < 2^{\partial^+} +$ the extra WdmId(∂) is not ∂^+-saturated" is needed, (if $\xi < \partial$ less is needed; however $\xi < \partial$ shall not be enough. But to get the sharp results (with the extra assumption) for almost good λ-frames we need a more elaborate approach - using vertical ξ-uq-invariant coding, see Definition 3.10.

Actually we shall use an apparently weaker version, the so called semi ξ-uq-invariant. However, we can derive from it the vertical version under reasonable demands on $\mathfrak{u}$; this last proof is of purely model theoretic characters. We also consider other variants.

In this section we usually do not use the τ from 3.1, i.e. use $\tau = \tau_{\mathfrak{u}}$ as it is not required presently.

3.1 Hypothesis. We assume $\mathfrak{u}$ is a nice construction framework and τ is a weak $\mathfrak{u}$-sub-vocabulary.

Remark. In Definition 3.2(1) below "$\mathbf{d}_*$ witnesses not being able to lift $\mathbf{d}$", of course we can ensure it can be lifted.

3.2 Definition. Let $\xi \leq \partial + 1$, if we omit it we mean $\xi = \partial + 1$.
1) We say that $(M, N, \mathbf{I}) \in \mathrm{FR}^1_{\mathfrak{u}}$ has weak ξ-uq-invariant coding$_0$ when:

- $\circledast$ if $M \leq_{\mathfrak{u}} M'$ and $M' \cap N = M$ then there are an ordinal $\alpha < \xi$ and a $\mathfrak{u}$-free $(\alpha, 0)$-rectangle $\mathbf{d}$ so $\alpha = \partial$ is O.K., such that:
  - (a) $M^{\mathbf{d}}_{0,0} = M'$ and $M^{\mathbf{d}}_{\alpha,0} \cap N = M$

(b) for every $N', \mathbf{I}'$ such that $(M', N', \mathbf{I}')$ is $\leq^1_{\mathfrak{u}}$-above $(M, N, \mathbf{I})$ and $N' \cap M^{\mathbf{d}}_{\alpha,0} = M'$ we can find $\alpha', \mathbf{I}^1, \alpha_*$ and $\mathbf{d}_*$ such that $\alpha' \leq \alpha, \alpha' < \partial$ (no harm[21] in $\alpha < \partial \Rightarrow \alpha' = \alpha$) and:

(α) $\mathbf{d}_*$ is a $\mathfrak{u}$-free $(\alpha_*, 0)$-rectangle and $\alpha_* < \partial$

(β) $M^{\mathbf{d}_*}_{0,0} = N'$ and $M^{\mathbf{d}}_{\alpha',0} \leq_{\mathfrak{u}} M^{\mathbf{d}_*}_{\alpha_*,0}$

(γ) $(M', N', \mathbf{I}') \leq^1_{\mathfrak{u}} (M^{\mathbf{d}}_{\alpha,0}, M^{\mathbf{d}_*}_{\alpha_*,0}, \mathbf{I}^1)$

(δ) there are no $\mathbf{d}_1, \mathbf{d}_2$ such that

- $\bullet_1$ $\mathbf{d}_\ell$ is a $\mathfrak{u}$-free rectangle for $\ell = 1, 2$
- $\bullet_2$ $\alpha(\mathbf{d}_1) = \alpha_*$ and $\mathbf{d}_1 \restriction (\alpha_*, 0) = \mathbf{d}_*$
- $\bullet_3$ $\alpha(\mathbf{d}_2) \geq \alpha'$ and $\mathbf{d}_2 \restriction (\alpha', 0) = \mathbf{d} \restriction (\alpha', 0)$
- $\bullet_4$ $(M^{\mathbf{d}_2}_{0,0}, M^{\mathbf{d}_2}_{0,1}, \mathbf{I}^{\mathbf{d}_2}_{0,0}) = (M', N', \mathbf{I}')$ or just $(M, N, \mathbf{I}) \leq^1_{\mathfrak{u}} (M^{\mathbf{d}_2}_{0,0}, M^{\mathbf{d}_2}_{0,1}, I^{\mathbf{d}_2}_{0,0})$
- $\bullet_5$ there are $N \in \mathfrak{K}_{\mathfrak{u}}$ and $\leq_{\mathfrak{u}}$-embeddings f_ℓ of $N^{\mathbf{d}_\ell}_{\alpha(\mathbf{d}_\ell),\beta(\mathbf{d}_\ell)}$ into N for $\ell = 1, 2$ such that $f_1 \restriction M^{\mathbf{d}}_{\alpha',0} = f_2 \restriction M^{\mathbf{d}}_{\alpha',0}$ and f_1, f_2 maps $M^{\mathbf{d}_1}_{0,\beta(\mathbf{d}_1)}$ onto $M^{\mathbf{d}_2}_{0,\beta(\mathbf{d}_2)}$.

2) We say that $(\bar{M}^*, \bar{\mathbf{J}}^*, \mathbf{f}^*) \in K^{\rm qt}_{\mathfrak{u}}$ has the weak ξ-uq-invariant coding$_0$ property <u>when</u>: if $\alpha(0) < \partial$ and $(M_*, N_0, \mathbf{I}_0) \in \mathrm{FR}^+_1, M_* \leq_{\mathfrak{u}} M^*_{\alpha(0)}$ and $N_0 \cap M^*_\partial = \emptyset$ <u>then</u> for some club E of ∂, for every $\delta \in E$ the statement $\circledast$ of part (1) holds, with $(M_*, N_0, \mathbf{I}_0), M_\delta$ here standing for $(M, N, \mathbf{I}), M'$ there but with some changes:

$(*)_1$ $\mathbf{d}$ is such that $M^{\mathbf{d}}_{\alpha',0} \leq_{\mathfrak{u}} M^*_\beta$ for each $\alpha' < \alpha$ for any $\beta < \partial$ large enough and

$(*)_2$ in clause (b) we demand $N' \cap M^*_\partial = M_\delta$ and $M^{\mathbf{d}_*}_{\alpha_*} \leq_{\mathfrak{u}} M^*_\beta$ for every $\beta < \partial$ large enough.

3) We say that $(\bar{M}^*, \bar{\mathbf{J}}^*, \mathbf{f}) \in K^{\rm qt}_{\mathfrak{u}}$ has the weak ξ-uq-invariant coding$_1$ property as in part (2) but require only that there are such $\alpha(0), (M_*, N_0, \mathbf{I}_0)$ so without loss of generality $M_* = M^*_{\alpha(0)}$.
3A) We say that $(\bar{M}, \bar{\mathbf{J}}, \mathbf{f})$ has the S-weak ξ-uq-invariant coding$_2$ property <u>when</u> we combine the above with Definition 2.6.
4) For $k = 0, 1$ we say that $(\bar{M}, \bar{\mathbf{J}}, \mathbf{f})$ has the S-weak ξ-uq-invariant coding$_k$ property <u>when</u>: S is a stationary subset of ∂ and for some club E of ∂ the demand in part (2) if $k = 0$, part (3) if $k = 1$ holds restricting ourselves to $\delta \in S \cap E$.

[21] by natural monotonicity, similarly in 3.7, 3.10, 3.14

5) We say that $\mathfrak{u}$ has the S-weak ξ-uq-invariant coding$_k$ property when: $\{0,2\}$-almost every $(\bar{M},\bar{\mathbf{J}},\mathbf{f}) \in K^{\rm qt}_{\mathfrak{u}}$ has the S-weak ξ-uq-invariant coding$_k$ property. Similarly for "above $(\bar{M}^*,\bar{\mathbf{J}}^*,\mathbf{f}^*) \in K^{\rm qt}_{\mathfrak{u}}$". If $S = \partial$ we may omit it; if $k=1$ we may omit it.

3.3 Claim. *Assume ($\xi \le \partial + 1$ and):*

- (a) *$S \subseteq \partial$ is stationary*
- (b) *$(\bar{M},\bar{\mathbf{J}},\mathbf{f}) \in K^{\rm qt}_{\mathfrak{u}}$ has the S-weak ξ-uq-invariant coding property*
- (c) *$\mathbf{f} \restriction S$ is constantly zero*
- (d) *$\xi \le \partial + 1$ and if $\xi = \partial$ then the ideal* $\mathrm{WDmId}(\partial) + (\lambda \backslash S)$ *is not ∂-saturated.*

Then we can find $\langle (\bar{N}^\eta,\bar{\mathbf{J}}^\eta,\mathbf{f}^\eta) : \eta \in {}^\partial 2\rangle$ such that

- (α) $(\bar{M},\bar{\mathbf{J}},\mathbf{f}) \le^{\rm qs}_{\mathfrak{u}} (\bar{N}^\eta,\bar{\mathbf{J}}^\eta,\mathbf{f}^\eta)$
- (β) $\mathbf{f}^\eta(\partial \backslash S) = \mathbf{f} \restriction (\partial \backslash S)$
- (γ) *if $\eta^1 \ne \eta^2 \in {}^\partial 2$ and $(\bar{N}^{\eta^\ell},\bar{\mathbf{J}}^{\eta^\ell},\mathbf{f}^{\eta^\ell}) \le^{\rm qt}_{\mathfrak{u}} (\bar{N}^\ell,\bar{\mathbf{J}}^\ell,\mathbf{f}^\ell)$ for $\ell = 1,2$ then $N^1_\partial, N^2_\partial$ are not isomorphic over M_∂.*

3.4 Remark. Note that in Definition 3.2(1) we choose the $\mathfrak{u}$-free $(\alpha_\delta,0)$-rectangle $\mathbf{d}_\delta$ for every $\delta \in S$ before we have arrived to choosing N^η_δ. This will be a burden in applying this.

Proof. For simplicity we first assume $\xi \le \partial$. Let $\langle S_\varepsilon : \varepsilon \le \partial\rangle$ be a sequence of pairwise disjoint stationary subsets of ∂ with union $S \backslash \{0\}$ such that $\varepsilon < \min(S_\varepsilon)$ (exists; if $\xi = \partial$ by assumption (d), otherwise if ∂ successor by applying Ulam matrixes, in general by a theorem of Solovay). Without loss of generality $0 \notin S$.

By assumption (b) we can find $\alpha(0) < \partial$ and $N_0,\mathbf{I}_0$ such that $N_0 \cap M_\partial = M_{\alpha(0)}$ and $(M_{\alpha(0)},N_0,\mathbf{I}) \in$ FR$_1$ and a club E_0 of ∂ such that and for every $\delta \in S \cap E_0$ we can choose $\alpha_\delta,\mathbf{d}_\delta$ as in $\circledast$ of Definition 3.2(1) with $M_{\alpha(0)},N_0,\mathbf{I}_0,M_\delta$ here standing for $M,N,\mathbf{I},M'$ there but demanding $M^{\mathbf{d}_\delta}_{\alpha_\delta,0} \le_{\mathfrak{u}} M_{\beta_\delta}$ for some $\beta_\delta \in (\delta,\partial)$, see $(*)_1$ of Definition 3.2(2). Without loss of generality $\alpha(0)$ is a successor ordinal.

Let

$$\begin{aligned} E_1 = \{\delta \in E_0 :& \delta > \alpha(0) \text{ and if } \delta(1) \in \delta \cap S \cap E_0 \\ & \text{then } \beta_{\delta(1)} \le \delta, \text{ i.e. } M^{\mathbf{d}_{\delta(1)}}_{\alpha_{\delta(1)},0} \le_{\mathfrak{u}} M_\delta \\ & \text{and } (\forall \beta < \delta)(\beta \times \beta < \delta \wedge \mathbf{f}(\beta) < \delta)\}. \end{aligned}$$

Clearly E_1 is a club of ∂.

We now choose $\langle(\delta^\rho, u^\rho, \bar N^\rho, \bar M^\rho, \bar{\mathbf{J}}^{0,\rho}, \bar{\mathbf{J}}^{1,\rho}, \mathbf{f}^\rho, \mathbf{I}^\rho, e^\rho) : \rho \in {}^i2\rangle$ by induction on $i < \partial$ such that

$\circledast$ (a) (α) $\delta^\rho < \partial$ belongs to $E_1 \cup \{\alpha(0)\}$

(β) e^ρ is a closed subset of $E_1 \cap \delta^\rho$

(γ) $\min(e^\rho) = \alpha(0)$, also $N^\rho_{\alpha(0)} = N_0, \mathbf{I}^\rho_{\alpha(0)} = \mathbf{I}$

(b) $\mathbf{f}^\rho : e^\rho \to \delta^\rho$

(c) $\mathbf{f}^\rho(\alpha) < \beta$ if $\alpha < \beta$ are from e^ρ

(d) if $\alpha \in e^\rho \backslash S$ then $\mathbf{f}^\rho(\alpha) = \mathbf{f}(\alpha)$

(e) (α) $\bar N^\rho = \langle N^\rho_i : i \le \delta^\rho\rangle$ and $\bar M^\rho = \langle M^\rho_i : i \le \delta^\rho\rangle$ are $\le_{\mathfrak{u}}$-increasing continuous

(β) $\langle (M^\rho_i, N^\rho_i, \mathbf{I}^{\rho\restriction i}) : i \le \delta\rangle$ is $\le^1_{\mathfrak{u}}$-increasing continuous

(f) (α) $\bar{\mathbf{J}}^{\ell,\rho} = \langle \mathbf{J}^{\ell,\rho}_i : i < \delta^\rho\rangle$

(β) $(M^\rho_i, M^\rho_{i+1}, \mathbf{J}^{0,\rho}_i) \in \mathrm{FR}_2$ for $i < \delta^\rho$

(γ) $(M^\rho_i, M^\rho_{i+1,\rho}, \mathbf{J}^{0,\rho}_i) \le^2_{\mathfrak{u}} (N^\rho_i, N^\rho_{i+1}, \mathbf{J}^{1,\rho}_i)$ when $i < \delta^\rho$ & $(\exists \alpha)(\alpha \in e^\rho \cap S$ & $\alpha \le i \le \alpha + \mathbf{f}^\rho(\alpha))$

(g) if $\alpha \in e^\rho$ then $M^\rho_\alpha = M_\alpha$ and $N_\alpha \cap M_\partial = M_\alpha$,

(h) if $\alpha \in e^\rho \backslash S$ then ($\mathbf{f}^\rho(\alpha) = \mathbf{f}(\alpha)$ and) $i \le \mathbf{f}(\alpha) \Rightarrow M^\rho_{\alpha+i} = M_{\alpha+i} \wedge N^\rho_{\alpha+i} \cap M_\partial = M_{\alpha+i} = M^\rho_{\alpha+i}$ and $i < \mathbf{f}(\alpha) \Rightarrow \mathbf{J}^{0,\rho}_{\alpha+i} = \mathbf{J}_{\alpha+i}$ and $i < \mathbf{f}(\alpha) \Rightarrow (M^\rho_{\alpha+i}, M^\rho_{\alpha+i+1}, \mathbf{J}^{0,\rho}_{\alpha+i}) \le^2_{\mathfrak{u}} (N^\rho_{\alpha+i}, N^\rho_{\alpha+i+1}, \mathbf{J}^{1,\rho}_{\alpha+i})$

(i) if $\varrho \triangleleft \rho$ then $\delta_\varrho < \delta_\rho, e^\varrho = \delta_\varrho \cap e^\rho, \bar M^\varrho \triangleleft \bar M^\rho, \bar N^\varrho \triangleleft \bar N^\rho, \bar{\mathbf{J}}^{\ell,\varrho} \triangleleft \bar{\mathbf{J}}^{\ell,\rho}$ for $\ell = 0, 1$ and $\bar{\mathbf{I}}^\varrho \triangleleft \bar{\mathbf{I}}^\rho$

(j) if $\varepsilon < \partial$ and $\alpha \in e^\rho \cap S$ and $\rho(\alpha) = 1$ then $(\langle M^\rho_{\alpha+i} : i \le \mathbf{f}^\rho(\alpha)\rangle, \langle \mathbf{J}^{0,\rho}_{\alpha+i} : i < \mathbf{f}^\rho(\alpha)\rangle)$ is equal to $\mathbf{d}_\alpha$

(k) if $\varepsilon < \partial, \alpha \in e^\rho \cap S$ and $\rho(\alpha) = 0$ then there is $\mathbf{d}_*$ as in clause (b) of $\circledast$ of Definition 3.2(1), for transparency of successor length β with $(M_{\alpha(0)}, N_0, \mathbf{I}_0), (M^\rho_\alpha, N^\rho_\alpha, \mathbf{I}^\rho_\alpha), \mathbf{d}_\alpha, \mathbf{d}_*$ here standing for $(M, N, \mathbf{I}), (M', N', \mathbf{I}'), \mathbf{d}, \mathbf{d}_*$ there, such that $\mathbf{f}^\rho(\alpha) = \alpha', N^\rho_{\alpha+i,0} = M^{\mathbf{d}_*}_i$ for $i \le \mathbf{f}^\rho(\alpha)$, $\mathbf{J}^{1,\rho}_{\alpha+i} = \mathbf{J}^{\mathbf{d}_*}_{i,0}$ for $i < \mathbf{f}^\rho(\alpha)$, and (for transparency) $M_{\alpha+i} = M_\alpha, \mathbf{J}^{0,\rho}_{\alpha+i} = \emptyset$ for $i \le \mathbf{f}^\rho(\alpha)$.

Why can we construct?

Case 1: $i = 0$

For $\rho \in {}^i 2$ let $\delta_\rho = \alpha(0)$ and by clause $(a)(\gamma)$ we define the rest (well for $i < \alpha(0)$ and $\ell = 0,1$ let $M^\ell_i = M_{\alpha(0)}, N^\rho_i = N_0, \mathbf{J}^{\ell,\rho}_i = \emptyset, \mathbf{I}^\rho_i = \mathbf{I}$).

Case 2: i is a limit ordinal

For $\rho \in {}^i 2$, let $\delta_\rho = \cup\{\delta_{\rho \restriction j} : j < i\}$ so $\delta_\rho \in E_1, \delta_\rho = \sup(E_1 \cap \alpha)$.

By continuity we can define also the others.

Case 3: $i = j+1$

Let $\rho \in {}^j 2$ and we define for $\rho\char`\^\langle \ell \rangle$ for $\ell = 0,1$ and first we deal only with $i \le \delta + \mathbf{f}^{\rho\char`\^<\ell>}(\delta)$.

Subcase 3A: $\delta_\rho \notin S$

We use clause (h) of $\circledast$ and 1.5(5).

Subcase 3B: $\delta \in S$

If $\ell = 1$ then we use $\mathbf{d}_\alpha$ for $\rho\char`\^\langle \ell \rangle$ as in clause (j) of $\circledast$ so the proof is as in subcase 3A. If $\ell = 0$ clause (b) of $\circledast$ of Definition 3.2 can be applied with $(M,N,\mathbf{I}),(M',N',\mathbf{I}'),\mathbf{d}$ there standing for $(M_{\alpha(0)},N_0,\mathbf{I}_0),(M^\rho_{\delta_\rho},N^\rho_{\delta_\rho},\mathbf{I}^\rho_{\delta_\rho}),\mathbf{d}_{\delta_\rho}$ here; so we can find $\alpha'_\rho, \mathbf{I}^1_\rho, \alpha^\rho_*, \mathbf{d}^\rho_*$ as there (presently α'_ρ there can be α_δ); and without loss of generality $\ell g(\mathbf{d}^\rho_*)$ is a successor ordinal.

Now we choose:

$(*)$ (a) $\mathbf{f}^{\rho\char`\^<0>}(\delta_\rho) = \ell g(\mathbf{d}^\rho_*)$

(b) $M^{\rho\char`\^<0>}_{\delta_\rho+i} = M^\rho_{\delta_\rho}$ for $i < \mathbf{f}^{\rho\char`\^<0>}(\delta)$

(c) $M^{\rho\char`\^<0>}_{\delta_\rho+i} = M^{\mathbf{d}_\delta}_{\alpha'_\rho,0}$ for $i = \mathbf{f}^{\rho\char`\^<0>}(\delta_\rho)$

(d) $N^{\rho\char`\^<0>}_{\delta_\rho+i} = N^{\mathbf{d}^\rho_*}_{i,0}$ for $i \le \mathbf{f}^{\rho\char`\^<0>}(\delta_\rho)$

(e) $\mathbf{J}^{0,\rho\char`\^<\ell>}_{\delta_\rho+i} = \emptyset, \mathbf{J}^{1,\rho\char`\^<\ell>}_{\delta_\rho+i} = \mathbf{J}^{\mathbf{d}^\rho_*}_{i,0}$ for $i < \mathbf{f}^{\rho\char`\^<0>}(\delta_\rho)$.

Clearly clause (k) holds. This ends the division to cases 3A,3B.

Lastly, choose $\delta_{\rho\char`\^<\ell>} \in E_1$ large enough; we still have to choose $(M^{\rho\char`\^<\ell>}_i, N^{\rho\char`\^<\ell>}_i, \mathbf{I}^{\rho\char`\^<\ell>})$ for $i \in (\delta_\rho + \mathbf{f}^\rho(\delta_\rho), \delta_{\rho\char`\^<\ell>}]$; we choose them all equal, $M^{\rho\char`\^<\ell>}_i = M_{\delta_{\rho\char`\^<\ell>}}$ and use 1.3(2) to choose $N^{\rho\char`\^<\ell>}_i, \mathbf{I}^{\rho\char`\^<\ell>}_i$. Then let $\mathbf{J}^{m,\rho\char`\^<\ell>}_i = \emptyset$ when $m < 2$ & $i \in [\delta_\rho + \mathbf{f}^\rho(\delta_\rho), \delta_{\rho\char`\^<\ell>})$.

So we have carried the induction. For $\rho \in {}^\partial 2$ we define $(\bar M^\rho, \bar{\mathbf{J}}^\rho, \mathbf{f}^\rho) \in K^{\rm qt}_{\mathfrak u}$ by $M^\rho_\alpha = N^{\rho \restriction i}_\alpha, \mathbf{J}^\rho_\alpha = \bar{\mathbf{J}}^{1,\rho\restriction i}_\alpha, \mathbf{f}^\rho(\alpha) = \mathbf{f}^{\rho\restriction i}(\alpha)$ for every $i < \partial$ large enough. Easily

$\odot_1$ $(\bar M, \bar{\mathbf{J}}, \mathbf{f}) \le^{\rm qt}_{\mathfrak u} (\bar M^\rho, \bar{\mathbf{J}}^\rho, \mathbf{f}^\rho)$.

Next

$\odot_2$ for $\nu \in {}^\partial 2$ let $\rho_\nu = \rho[\nu] \in {}^\partial 2$ be defined by $\rho_\nu(i) = \nu(\varepsilon)$ if $i \in S_\varepsilon \wedge \varepsilon < \partial$ and zero otherwise.

So it is enough to prove

$\odot_3$ if $\nu_1 \neq \nu_2 \in {}^{\partial}2$ and $(\bar{M}^{\rho[\nu_\ell]}, \bar{\mathbf{J}}^{\rho[\nu_\ell]}, \mathbf{f}^{\rho[\nu_\ell]}) \leq^{\rm qt}_{\mathfrak{u}} (\bar{M}^\ell, \bar{\mathbf{J}}^\ell, \mathbf{f}^\ell)$ for $\ell = 1,2$, then $M^1_\partial, M^2_\partial$ are not isomorphic over M_∂.

Why this holds? As $\nu_1 \neq \nu_2$, by symmetry without loss of generality for some $\varepsilon < \partial$ we have $\nu_1(\varepsilon) = 1, \nu_2(\varepsilon) = 0$, and let f be an isomorphism from M^1_∂ onto M^2_∂ over M_∂ and let E^ℓ witness $(\bar{M}^{\rho[\nu_\ell]}, \bar{\mathbf{J}}^{\rho[\nu_\ell]}, \mathbf{f}^{\rho[\nu_\ell]}) \leq^{\rm qt}_{\mathfrak{u}} (\bar{M}^\ell, \bar{\mathbf{J}}^\ell, \mathbf{f}^\ell)$ for $\ell = 1,2$. Let $E := E^1 \cap E^2 \cap \{\delta < \partial : \delta \in \cup\{e^{\rho[\nu_\ell] \restriction i} : i < \partial\}$ and $\delta = \text{ otp}(\delta \cap e_{\rho[\nu_\ell] \restriction \delta})$ for $\ell = 1,2$ and f maps M^1_δ onto $M^2_\delta\}$, clearly it is a club of ∂.

Hence there is $\delta \in S_\varepsilon \cap E$, so $\rho[\nu_1](\delta) = 1, \rho[\nu_2](\delta) = 0$, now the contradiction is easy, recalling:

$(*)_1$ clause (δ) of Defintion 3.2(1)

$(*)_2$ $\mathbf{d}_\delta$ does not depend on $\rho[\nu_\ell] \restriction \delta$.

We still owe the proof in the case $\xi = \partial + 1$, it is similar with two changes. The first is in the choice of $\mathbf{d}_\delta$, as now $\alpha_{\mathbf{d}_\delta}$ may be ∂, so β_δ may be ∂, hence we have to omit "$\beta_{\delta(1)} \leq \delta$" in the definition of E_1. In clause $\circledast(j),(k)$ we should replace $\mathbf{d}_\alpha$ by $\mathbf{d}_\alpha \restriction (0, \gamma_{\rho \restriction \alpha})$ so $\alpha + \gamma_{\rho \restriction \delta} < \text{ min}(e_\rho \backslash (\alpha+1))$ where $\gamma_{\rho \restriction \alpha} < \text{ min}\{\ell g(\mathbf{d}_\alpha) + 1, \partial\}$ which is the minimal α when we apply α in Definition 3.2.

Second, the choice of $\langle \rho_\nu : \nu \in {}^{\partial}2 \rangle$ with $\rho_\nu \in {}^{\partial}2$ is more involved.

For each $\varepsilon < \partial$ we choose $\varrho_\varepsilon \in {}^{(S_\varepsilon)}2$ such that:

$\boxdot$ if $\rho_1, \rho_2 \in {}^{\partial}2$ then for stationarily many $\delta \in S_\varepsilon$ we have: $\mathbf{f}^{\rho_1}(\delta) \leq \mathbf{f}^{\rho_2}(\delta) \Leftrightarrow \varrho_\varepsilon(\delta) = 1$

(noting that $\mathbf{f}^{\rho_\alpha}(\delta)$ depends only on $\rho \restriction \delta$).

[Why possible? As S_ε is not in the weak diamond ideal.]

Then we replace $\odot_2$ by

$\odot'_2$ for $\nu \in {}^{\partial}2$ let $\rho_\nu = \rho[\nu] \in {}^{\partial}2$ be defined by $\rho_\nu(i) = \varrho_\varepsilon(i) + \nu(\varepsilon)$ mod 2 when $i \in S_\varepsilon \wedge \varepsilon < \partial$ and $\rho_\nu(i)$ is zero otherwise.

Why this is O.K.? I.e. we have to prove $\odot_3$ in this case. Why this holds? As $\nu_1 \neq \nu_2$ by symmetry without loss of generality $\nu_1(\varepsilon) = 1, \nu_2(\varepsilon) = 0$ and let f, E be as before.

Let $\rho_\ell := \rho[\nu_\ell] \in {}^{\partial}2$, so by the choice of ϱ_ε there is $\delta \in E \cap S_\varepsilon$ such that

$$\mathbf{f}^{\rho_1}(\delta) \leq \mathbf{f}^{\rho_2}(\delta) \Leftrightarrow \varrho_\varepsilon(\delta) = 1.$$

First assume $\varrho_\varepsilon(\delta) = 1$ hence $\mathbf{f}^{\rho_1}(\delta) \leq \mathbf{f}^{\rho_2}(\delta)$, so $\rho_1(\delta) = \varrho_\varepsilon(\delta) + \nu_1(\varepsilon) = 1 + 1 = 0$ mod 2 so $\rho_1(\delta) = 0$ and $\rho_2(\delta) = \varrho_\varepsilon(\delta) + \nu_2(\varepsilon) = 1 + 0 = 1$ mod 2, so $\nu_2(\delta) = 1$.

Now we continue as before, because what we need there for $(\bar M^{\rho[\nu_2]}, \bar{\mathbf{J}}^{\rho[\nu_2]}, \mathbf{f}^{\rho[\nu_2]})$ in δ is satisfied for $\delta + \mathbf{f}^{\nu[\rho_2]}(\delta)$ hence also for $\delta + \mathbf{f}^{\nu[\rho_1]}$.

The other case, $\varrho_\varepsilon(\delta) = 0$, is similar; exchanging the roles. $\square_{3.3}$

The following conclusion will be used in 6.15, 6.16.

3.5 Conclusion. We have $\dot I(\partial^+, \mathfrak{K}) \geq \mu_{\text{unif}}(\partial^+, 2^\partial)$ and moreover $\dot I(K^{\mathfrak{u},\mathfrak{h}}_{\partial^+}) \geq \mu_{\text{unif}}(\partial^+, 2^\partial)$ for any $\{0,2\}$-appropriate $\mathfrak{h}$ <u>when</u> ($\xi \leq \partial + 1$ and):

- (a) $2^\partial < 2^{\partial^+}$ and $\partial > \aleph_0$
- (b) (α) $\mathscr{D}_\partial$, the club filter on ∂, is not ∂^+-saturated
  - (β) if $\xi = \partial + 1$ then $\text{WDmId}(\partial) + (\partial \backslash S)$ is not ∂^+-saturated
- (c) $\{0,2\} - S$-almost every $(\bar M, \bar{\mathbf{J}}, \mathbf{f})$ has the weak ξ-uq-invariant coding property even just above $(\bar M^*, \bar{\mathbf{J}}, \mathbf{f})$, so $S \subseteq \partial$ is stationary.

Proof. By 3.3 we can apply 3.6 below using

$\odot$ if $\mathscr{D}$ is a normal filter on the regular uncountable $\partial, \mathscr{D}$ not ∂^+-saturated then also the normal ideal generated by $\mathscr{A}$ is not ∂^+-saturated where $\mathscr{A} = \{A \subseteq \partial : A \in \mathscr{D}$ or $\partial \backslash A \in \mathscr{A}^+$ but $\mathscr{D} + (\partial \backslash A)$ is ∂-saturated$\}$.

$\square_{3.5}$

3.6 Theorem. *1) We have $\dot I(\partial^+, K^{\mathfrak{u}}_{\partial^+}) \geq \mu_{\text{unif}}(\partial^+, 2^\partial)$; moreover $\dot I(K^{\mathfrak{u},\mathfrak{h}}_{\partial^+}) \geq \mu_{\text{unif}}(\partial^+, 2^\partial)$ for any $\mathfrak{u} - \{0,2\}$-appropriate $\mathfrak{h}$ (see Definition 1.24) <u>when</u>:*

- (a) $2^\partial < 2^{\partial^+}$
- (b) *$\mathscr{D}$ is a non-∂^+-saturated normal filter on ∂*
- (c) *for $\{0,2\}$-almost every $(\bar M, \bar{\mathbf{J}}, \mathbf{f}) \in K^{\text{qt}}_{\mathfrak{u}}$ (maybe above some such triple $(\bar M^*, \bar{\mathbf{J}}^*, \mathbf{f}^*)$ satisfying $\mathscr{D}_\partial + \mathbf{f}^{-1}\{0\}$ is not ∂^+-saturated), if $S \subseteq \partial$ belongs to $\mathscr{D}^+$ and $\mathbf{f} \restriction S$ is constantly zero then we can find a sequence $\langle (\bar M^\alpha, \bar{\mathbf{J}}^\alpha, \mathbf{f}^\alpha) : \alpha < 2^\partial \rangle$ such that*
  - (α) $(\bar M, \bar{\mathbf{J}}, \mathbf{f}) \leq^{\text{qt}}_{\mathfrak{u}} (\bar M^\alpha, \bar{\mathbf{J}}^\alpha, \mathbf{f}^\alpha)$ *and* $\mathbf{f}^\alpha \restriction (\partial \backslash S) = \mathbf{f} \restriction (\partial \backslash S)$
  - (β) *if $\alpha(1) \neq \alpha(2) < 2^\partial$ and $(\bar M^{\alpha(i)}, \bar{\mathbf{J}}^{\alpha(i)}, \mathbf{f}^{\alpha(i)}) \leq^{\text{qt}}_{\mathfrak{u}} (\bar M^{\ell,*}, \bar{\mathbf{J}}^{\ell,*}, \mathbf{f}^{\ell,*})$ for $\ell = 1,2$ then $M^{1,*}_\partial, M^{2,*}_\partial$ are not isomorphic over M_∂.*

2) Similarly omitting the "∂^+-saturation" demands in clauses (b),(c) and omitting $\mathbf{f} \restriction S$ *is constantly zero in clause (c).*

Proof. 1) By Observation 1.25(4) without loss of generality $\mathfrak{h} = \mathfrak{h}_0 \cup \mathfrak{h}_2$ witness clause (c) of the assumption; we shall use $\mathfrak{h}_2$ for S^*_0 so without loss of generality $\mathfrak{h}_2$ is a $2 - S^*_0$-appropriate. By clause (b) of the assumption let $\bar{S}^*$ be such that

$\odot$ (a) $\bar{S}^* = \langle S^*_\alpha : \alpha < \partial^+ \rangle$
(b) $S^*_\alpha \subseteq \partial$ for $\alpha < \partial^+$
(c) $S^*_\alpha \backslash S^*_\beta \in [\partial]^{<\partial}$ for $\alpha < \beta < \partial^+$
(d) $S^*_{\alpha+1} \backslash S^*_\alpha$ is a stationary subset of ∂; moreover $\in \mathscr{D}^+$
(e) $S = S^*_0$ is stationary; it includes $\{\delta < \partial : \mathbf{f}^*(\delta) > 0\}$ when $(\bar{M}^*, \bar{\mathbf{J}}^*, \mathbf{f}^*)$ is given.

Let $(\bar{M}^*, \bar{\mathbf{J}}^*, \mathbf{f}^*)$ be as in clause (c) of the assumption; so $\{0,2\}$-almost every $(\bar{M}^{**}, \bar{\mathbf{J}}^{**}, \mathbf{f}^{**}) \in K^{\rm qt}_{\mathfrak{u}}$ which is $\leq_{\rm qs}$-above $(\bar{M}^*, \mathbf{J}^*, \mathbf{f}^*)$ is as there witnessed by $\mathfrak{h}$.

Now we choose $\langle (\bar{M}^\eta, \bar{\mathbf{J}}^\eta, \mathbf{f}^\eta) : \eta \in {}^\alpha(2^\partial) \rangle$ by induction on $\alpha < \partial^+$ such that

$\circledast$ (a) $(\bar{M}^\eta, \bar{\mathbf{J}}^\eta, \mathbf{f}^\eta) \in K^{\rm qt}_{\mathfrak{u}}$, and is equal to $(\bar{M}^*, \mathbf{J}^*, \mathbf{f}^*)$ if $\eta = <>$
(b) $\langle (\bar{M}^{\eta \restriction \beta}, \bar{\mathbf{J}}^{\eta \restriction \beta}, \mathbf{f}^{\eta \restriction \beta}) : \beta \leq \ell g(\eta) \rangle$ is $\leq^{\rm qs}_{\mathfrak{u}}$-increasing continuous
(c) $\mathbf{f}^\eta \restriction (\partial \backslash S^*_{\ell g(\eta)+1})$ is constantly zero
(d) if $\ell g(\eta) = \beta + 2 \leq \alpha$ and $\nu = \eta \restriction (\beta+1))$ then $(\bar{M}^\eta, \bar{\mathbf{J}}^\eta, \mathbf{f}^\eta) \leq^{\rm at}_{\mathfrak{u}} (\bar{M}^\nu, \bar{\mathbf{J}}^\nu, \mathbf{f}^\nu)$ and this pair strictly S-obeys $\mathfrak{h}$
(e) if $\ell g(\eta) = \delta < \alpha, \delta$ limit or zero, $\varepsilon^0 < \varepsilon^1 < 2^\partial$ and $(\bar{M}^{\eta\hat{}<\varepsilon^\ell>}, \bar{\mathbf{J}}^{\eta\hat{}<\varepsilon^\ell>}, \mathbf{f}^{\eta\hat{}<\varepsilon^\ell>}) \leq^{\rm qt}_{\mathfrak{u}} (\bar{M}^\ell, \bar{\mathbf{J}}^\ell, \mathbf{f}^\ell)$ for $\ell = 0,1$ <u>then</u> $M^1_\partial, M^2_\partial$ are not isomorphic over M^η_∂.

The inductive construction is straightforward:
if $\alpha = 0$ let $(\bar{M}^\eta, \bar{\mathbf{J}}^\eta, \mathbf{f}^\eta) = (\bar{M}^*, \bar{\mathbf{J}}^*, \mathbf{f}^*)$
if α is limit use claim 1.19(4)
if $\alpha = \beta + 2$ use clause $\circledast(d)$
if $\alpha = \delta + 1, \delta$ limit or zero use clause (c) of the assumption to satisfy clause $\circledast(e)$.

Having carried the induction, let $M_\eta = \cup\{M^{\eta \restriction \alpha} : \alpha < \partial^+\}$ for $\eta \in {}^{\partial^+}(2^\partial)$. By 9.1 we get that $|\{M^\eta / \cong : \eta \in {}^{\partial^+}(2^\partial)\}| \geq \mu_{\rm unif}(\partial^+, 2^\partial)$ so we are done.
2) Similarly. $\square_{3.5}$

* * *

We now note how we can replace the ξ-uq-invariant by ξ-up-invariant, a relative, not used.

3.7 Definition. Let $\xi \le \partial + 1$.
1) We say that $(M, N, \mathbf{I}) \in \mathrm{FR}^1_{\mathfrak{u}}$ has the weak ξ-up-invariant coding property <u>when</u>:

- $\circledast$ if $M \le_{\mathfrak{u}} M'$ and $M' \cap N = \emptyset$ <u>then</u> there are $\alpha_\ell < \xi$ and $\mathfrak{u}$-free $(\alpha_\ell, 0)$-rectangle $\mathbf{d}_\ell$ for $\ell = 1, 2$ such that:
  - (a) $M^{\mathbf{d}_1}_{0,0} = M' = M^{\mathbf{d}_2}_{0,0}$
  - (b) $M^{\mathbf{d}_1}_{\alpha_1,0} = M^{\mathbf{d}_2}_{\alpha_2,0}$
  - (c) $M^{\mathbf{d}_\ell}_{\alpha_\ell,0} \cap N = M$
  - (d) if $(M, N, \mathbf{I}) \le_1 (M', N', \mathbf{I}')$ and $M^{\mathbf{d}_\ell}_{\alpha_\ell,0} \cap N' = M'$ <u>then</u> there are no $\alpha'_\ell \le \alpha_\ell, \alpha'_\ell < \partial, \beta_\ell < \partial$ and $\mathfrak{u}$-free $(\alpha'_\ell, \beta_\ell)$-rectangles $\mathbf{d}^\ell$ for $\ell = 1, 2$ such that
    - $\bullet_1$ $\mathbf{d}^\ell \restriction (\alpha_\ell, 0) = \mathbf{d}_\ell \restriction (\alpha'_\ell, 0)$
    - $\bullet_2$ $(M', N', \mathbf{I}') \le_1 (M^{\mathbf{d}^\ell}_{0,0}, M^{\mathbf{d}^\ell}_{0,1}, \mathbf{I}^{\mathbf{d}^\ell}_{\alpha,0})$
    - $\bullet_3$ there are N'', f such that $M^{\mathbf{d}^2}_{\alpha'_2,\beta_2} \le_{\mathfrak{u}} N$ and f is a $\le_{\mathfrak{u}}$-embedding of $N^{\mathbf{d}^1}_{\alpha'_1,\beta_1}$ into N over $M^{\mathbf{d}_1}_{\alpha'_1,0} = M^{\mathbf{d}_2}_{\alpha_2,0}$ mapping $M^{\mathbf{d}^1}_{0,\beta_1}$ onto $M^{\mathbf{d}^2}_{0,\beta_2}$.

2)-5) As in Definition 3.2 replacing uq by up.

3.8 Claim. *Like 3.3 replacing uq-invariant by up-invariant.*

Proof. Similar. $\square_{3.8}$

3.9 Conclusion. Like 3.5 replacing uq-invariant by up-invariant (in clause (c)).

Proof. Similar. $\square_{3.11}$

* * *

Another relative is the vertical one.

3.10 Definition. Let $\xi \le \partial + 1$, omitting ξ means $\partial + 1$. We say that $(M, N, \mathbf{I}) \in \mathrm{FR}^1_{\mathfrak{u}}$ has the vertical ξ-uq-invariant coding$_1$ property <u>when</u>:

$\otimes$ if $\alpha_0 < \partial$ and $\mathbf{d}_0$ is an $\mathfrak{u}$-free $(\alpha_0, 0)$-rectangle satisfying $M \le_{\mathfrak{u}} M^{\mathbf{d}_0}_{0,0}$ and $M^{\mathbf{d}_0}_{\alpha_0,0} \cap N = M$ <u>then</u> there are $\alpha, \mathbf{d}$ such that:

(a) $\alpha_0 < \alpha < \xi$

(b) $\mathbf{d}$ is a $\mathfrak{u}$-free $(\alpha, 0)$-rectangle, though α is possibly ∂ this is O.K.

(c) $\mathbf{d} \restriction (\alpha_0, 0) = \mathbf{d}_0$

(d) $M^{\mathbf{d}}_{\alpha,0} \cap N = M$

(e) for every $N', \mathbf{I}'$ such that $(M^{\mathbf{d}}_{0,0}, N', \mathbf{I}') \in$ FR_1 is $\le^1_{\mathfrak{u}}$-above $(M, N, \mathbf{I})$ and $N' \cap M^{\mathbf{d}}_{\alpha,0} = M^{\mathbf{d}}_{0,0}$ we can find $\alpha', \alpha_*, \mathbf{d}_*, \mathbf{I}'', M''$ such that

(α) $\alpha' \le \alpha, \alpha' < \partial, \alpha_0 \le \alpha_*$

(β) $\mathbf{d}_*$ is a $\mathfrak{u}$-free $(\alpha_*, 0)$-rectangle

(γ) $M^{\mathbf{d}_*}_{0,0} = N'$

(δ) there is an $\mathfrak{u}$-free $(\alpha_0, 1)$-rectangle $\mathbf{d}'$ such that $\mathbf{d}' \restriction (\alpha_0, 0) = \mathbf{d}_0, \mathbf{d}' \restriction ([0, \alpha_0), [1, 1]) = \mathbf{d}_* \restriction (\alpha_0, 0)$ and $\mathbf{I}^{\mathbf{d}'}_{0,0} = \mathbf{I}'$ and $(M^{\mathbf{d}'}_{\alpha_0,0}, M^{\mathbf{d}'}_{\alpha_0,1}, \mathbf{I}') \le^1_{\mathfrak{u}} (M'', M^{\mathbf{d}_*}_{\alpha_*,0}, \mathbf{I}'')$

(ε) there are no $\mathbf{d}_1, \mathbf{d}_2$ such that

$\bullet_1$ $\mathbf{d}_\ell$ is a $\mathfrak{u}$-free rectangle for $\ell = 1, 2$

$\bullet_2$ $\alpha(\mathbf{d}_1) = \alpha_*$ and $\mathbf{d}_1 \restriction (\alpha_*, 0) = \mathbf{d}_*$

$\bullet_3$ $\alpha(\mathbf{d}_2) \ge \alpha'$ and $\mathbf{d}_2 \restriction (\alpha', 0) = \mathbf{d} \restriction (\alpha', 0)$

$\bullet_4$ $(M^{\mathbf{d}_2}_{0,0}, M^{\mathbf{d}_2}_{0,1}, \mathbf{I}^{\mathbf{d}_2}_{0,0}) = (M^{\mathbf{d}}_{0,0}, N', \mathbf{I}')$ or just $(M^{\mathbf{d}}_{0,0}, N', \mathbf{I}') \le^1_{\mathfrak{u}} (M^{\mathbf{d}_2}_{0,0}, M^{\mathbf{d}_2}_{0,1}, \mathbf{I}^{\mathbf{d}_2}_{0,0})$

$\bullet_5$ there are N_1, N_2, f such that $M^{\mathbf{d}_\ell}_{0,\beta(\mathbf{d}_\ell)} \le_{\mathfrak{u}} N_\ell$ for $\ell = 1, 2$ and f is an isomorphism from N_1 onto N_2 over $M^{\mathbf{d}}_{\alpha,0}$ mapping $M^{\mathbf{d}_1}_{0,\beta(\mathbf{d}_1)}$ onto $M^{\mathbf{d}_2}_{0,\beta(\mathbf{d}_2)}$.

2) We say that $(\bar{M}^*, \bar{\mathbf{J}}^*, \mathbf{f}^*) \in K^{\mathrm{qt}}_{\mathfrak{u}}$ has the vertical uq-invariant coding$_1$ propety as in Definition 3.2(2) only $(\langle M_{\delta+i} : i \le \mathbf{f}(\delta)\rangle, \langle \mathbf{J}_{\delta+i} : i < \mathbf{f}(\delta)\rangle)$ play the role of $\mathbf{d}_0$ in part (1). In all parts coding means coding$_1$.
3),4),5) Parallely to Definition 3.2.

3.11 Theorem. *Like 3.3 using vertical ξ-uq-invariant coding in clause (b) and omitting clause (c) of the assumption and omit clause (β) in the conclusion.* $\square_{3.11}$

Proof. Similarly.

3.12 Conclusion. Like 3.5 replacing clause (b)(β) of the assumption (by $\xi = \partial+1 \Rightarrow (\exists\theta)2^\theta = 2^{<\partial} < 2^\partial$) and with vertical ξ-uq-invariant coding instead of the ξ-uq-invariant one (in clause (c), can use $S = \partial$).

Proof. Similar to the proof of 3.5. $\square_{3.12}$

* * *

<u>3.13 Discussion</u>: The intention below is to help in §6 to eliminate the assumption "WDmId(λ^+) is not λ^{++}-saturated" when $\mathfrak{s}$ fails existence for $K^{3,\mathrm{up}}_{\mathfrak{s},\lambda^+}$. We do using the following relatives, semi and vertical, from Definition 3.14, 3.10 are interesting because

- (a) under reasonable conditions (see Definition 3.17) the first implies the second
- (b) the second, as in Theorem 2.11 is enough for non-structure without the demand on saturation of WDmId(∂)
- (c) the first needs a weak version of a model theoretic assumption (in the application)
- (d) (not used) the semi-version implies the weak version (from 3.2).

3.14 Definition. Let $\xi \le \partial + 1$.
1) We say that $\mathfrak{u}$ has the semi ξ-uq-invariant coding$_1$ property [above some $(\bar M^*, \bar{\mathbf{J}}^*, \mathbf{f}^*)$] <u>when</u> for $\{0,2\}$-almost every $(\bar M, \bar{\mathbf{J}}, \mathbf{f}) \in K^{\mathrm{qt}}_{\mathfrak{u}}$ [above $(\bar M^*, \bar{\mathbf{J}}^*, \mathbf{f}^*)$] for some $(\alpha, N, \mathbf{I})$ we have $\alpha < \partial, M \cap N = M_\alpha$ and $(M_\alpha, N, \mathbf{I}) \in \mathrm{FR}^+_1$ has the semi ξ-uq-invariant coding$_1$ property, see below but restricting ourselves to $M', M^{\mathbf{d}}_{\alpha_{\mathbf{d}},0} \in \{M_\beta : \beta \in (\alpha, \partial)\}$. Here and in part (2) we may write coding instead of coding$_1$.
2) We say that $(M, N, \mathbf{I}) \in \mathrm{FR}^1_{\mathfrak{u}}$ has the semi ξ-uq-invariant coding$_1$ property (we may omit the 1) <u>when</u>: if $M \le_{\mathfrak{u}} M'$ and $M' \cap N = M$ <u>then</u> we can find $\mathbf{d}$ such that:

$\circledast$ (a) $\mathbf{d}$ is a $\mathfrak{u}$-free $(\alpha_{\mathbf{d}}, 0)$-rectangle with $\alpha_{\mathbf{d}} < \xi$ so $\alpha_{\mathbf{d}} \le \partial$

(b) $M^{\mathbf{d}}_{0,0} = M'$ and $M^{\mathbf{d}}_{\alpha(\mathbf{d}),0} \cap N = M$

(c) for any $N', \mathbf{I}'$; if $(M, N, \mathbf{I}) \le_1 (M', N', \mathbf{I}')$ and $M^{\mathbf{d}}_{\alpha(\mathbf{d}),0} \cap N' = M'$
<u>then</u> we can find $\alpha', N'', \mathbf{I}''$ satisfying $\alpha' \le \alpha, \alpha' < \partial$ and $(M', N', \mathbf{I}') \le_1 (M^{\mathbf{d}}_{\alpha',0}, N'', \mathbf{I}'') \in \mathrm{FR}^1_{\mathfrak{u}}$ such that for no triple $(\mathbf{e}, f, N_*)$ do we have:

(α) $\mathbf{e}$ is a $\mathfrak{u}$-free rectangle

(β) $\alpha_{\mathbf{e}} = \alpha_{\mathbf{d}}$ and $\mathbf{e} \restriction (\alpha', 0) = \mathbf{d} \restriction (\alpha', 0)$

(γ) $(M^{\mathbf{e}}_{0,0}, M^{\mathbf{e}}_{0,1}, \mathbf{I}^{\mathbf{e}}_{0,0}) = (M^{\mathbf{d}}_{0,0}, N'', \mathbf{I}'')$

(δ) $M^{\mathbf{e}}_{\alpha',\beta(\mathbf{e})} \le_{\mathfrak{u}} N_*$

(ε) f is a $\le_{\mathfrak{u}}$-embedding of N'' into N_*

(ζ) $f \restriction M^{\mathbf{d}}_{\alpha',0}$ is the identity

(η) f maps N' into $M^{\mathbf{e}}_{0,\beta(\mathbf{e})}$.

3.15 Remark. 1) This is close to Definition 3.2 but simpler, cover the applications here and fit Claim 3.20.
2) We could have phrased the other coding properties similarly.

3.16 Claim. *If $(M, N, \mathbf{I}) \in \mathrm{FR}^1_{\mathfrak{u}}$ has the semi ξ-uq-invariant coding property <u>then</u> it has the weak ξ-uq-invariant coding property, see Definition 3.2.*

Proof. Should be clear. $\square_{3.16}$

The following holds in our natural examples when we add the fake, i.e. artificial equality and it is natural to demand $\mathfrak{u} = \mathfrak{u}^{[*]}$, see Definition 3.18.

3.17 Definition. 1) We say that $\mathfrak{u}$ has the fake equality $=_*$ <u>when</u>:

(a) $\tau_{\mathfrak{K}}$ has only predicates and some two-place relation $=_* \in \tau_{\mathfrak{K}}$ is, for every $M \in K$, interpreted as an equivalence relation which is a congruence relation on M

(b) $M \in K$ iff $M/=^M_*$ belongs to K

(c) for $M \subseteq N$ both from K we have $M \le_{\mathfrak{u}} N$ iff $(M/=^N_*) \le_{\mathfrak{s}} (N/=^N_*)$

(d) assume $M \le_{\mathfrak{u}} N$ and $\mathbf{I} \subseteq N \backslash M$ and $\mathbf{I}' = \{d \in \mathbf{I} : (\forall c \in M)(\neg c =^N d)\}, \ell \in \{1,2\}$. If $(M, N, \mathbf{I}) \in \mathrm{FR}_\ell$ then $(M/=^N, N/=^N, \mathbf{I}'/=^N) \in \mathrm{FR}_\ell$ which implies $(M, N, \mathbf{I}') \in \mathrm{FR}_\ell$

(e) if $M \subseteq N$ are from K and $\mathbf{I} \subseteq \{d \in N : (\forall c \in M)(\neg c =^N d)\}$ and $\ell \in \{1,2\}$ <u>then</u> $(M, N, \mathbf{I}) \in \mathrm{FR}_\ell$ iff $(M/=^N, \mathbf{I}/=^N) \in \mathrm{FR}_\ell$.

1A) In part (1) we may say that $\mathfrak{u}$ has the fake equality $=_*$ or $=_*$ is a fake equality for $\mathfrak{u}$.
2) We say $\mathfrak{u}$ is hereditary <u>when</u> every $(M, N, \mathbf{I}) \in \mathrm{FR}^+_1$ is hereditary, see below.
3) We say $(M, N, \mathbf{I}) \in \mathrm{FR}^1_{\mathfrak{u}}$ is hereditary <u>when</u>:

- (a) if $\mathbf{d}$ is $\mathfrak{u}$-free $(1,2)$-rectangle, $M \le_{\mathfrak{u}} M^{\mathbf{d}}_{0,0}$ and $(M, N, \mathbf{I}) \le^1_{\mathfrak{u}} (M^{\mathbf{d}}_{0,1}, M^{\mathbf{d}}_{0,2}, \mathbf{I}^{\mathbf{d}}_{0,1})$ <u>then</u> $(M, N, \mathbf{I}) \le^1_{\mathfrak{u}} (M^{\mathbf{d}}_{0,0}, M^{\mathbf{d}}_{0,2}, \mathbf{I}^{\mathbf{d}}_{0,1}) \le^1_{\mathfrak{u}} (M^{\mathbf{d}}_{1,0}, M^{\mathbf{d}}_{1,2}, \mathbf{I}^{\mathbf{d}}_{1,1})$.

4) We say $\mathfrak{u}$ is hereditary for the fake equality $=_*$ when every $(M, N, \mathbf{I}) \in \mathrm{FR}^+_1$ is hereditary for $=_*$ which means that clause (a) of part (3) above holds, $=_*$ is a fake equality for $\mathfrak{u}$ and:

- (b) if $\mathbf{d}$ is a $\mathfrak{u}$-free $(0,2)$-rectangle, $M \le_{\mathfrak{u}} M^{\mathbf{d}}_{0,0}$ and $(M, N, \mathbf{I}) \le^1_{\mathfrak{u}} (M^{\mathbf{d}}_{0,1}, M^{\mathbf{d}}_{0,2}, \mathbf{I}^{\mathbf{d}}_{0,1})$ <u>then</u> we can find M_1, f, M_2 such that:
  - (α) f is an isomorphism from $M^{\mathbf{d}}_{0,1}$ onto M_1 over $M^{\mathbf{d}}_{0,0}$
  - (β) $M^{\mathbf{d}}_{0,0} \le_{\mathfrak{u}} M_1 \le_{\mathfrak{u}} M_2$ and $M_{0,2} \le_{\mathfrak{u}} M_2$
  - (γ) $|M_2| = |M_1| \cup |M^{\mathbf{d}}_{0,2}|$
  - (δ) $M_2 \models$ "$c =_* f(c)$" if $c \in M_{0,1}$
  - (ε) $(M_{0,0}, M_{0,2}, \mathbf{I}^{\mathbf{d}}_{0,1}) \le^1_{\mathfrak{u}} (M_1, M_2, \mathbf{I}^{\mathbf{d}}_{0,1})$.

5) In parts (2),(3),(4) we can replace hereditary by weakly hereditary <u>when</u>: in clause (a) we assume $\mathbf{I} = \mathbf{I}^{\mathbf{d}}_{0,1} = \mathbf{I}^{\mathbf{d}}_{1,1}$ and in clause (b) we assume $\mathbf{I} = \mathbf{I}^{\mathbf{d}}_{0,1}$.

3.18 Definition. For $\mathfrak{u}$ is a nice construction framework we define $\mathfrak{u}^{[*]} = \mathfrak{u}^{[*]}$ like $\mathfrak{u}$ except that, for $\ell = 1, 2$ we have $(M_1, N_1, \mathbf{I}_1) \le^\ell_{\mathfrak{u}[*]} (M_2, N_2, \mathbf{I}_2)$ <u>iff</u> $(M_1, N_1, \mathbf{I}_1) \le^\ell_{\mathfrak{u}} (M_2, N_2, \mathbf{I}_2)$ and $\mathbf{I}_1 \ne \emptyset \Rightarrow \mathbf{I}_1 = \mathbf{I}_2$.

3.19 Observation. 1) $\mathfrak{u}$ has the fake equality $=$ (i.e. the standard equality is also a fake equality).
2) $\mathfrak{u}'$ as defined in 2.8(2) has the fake equality $=_\tau$ (and is a nice construction framework, see 2.8(3)).
3) If $\mathfrak{u}$ is hereditary <u>then</u> $\mathfrak{u}''$ as defined in 2.8(4) is hereditary and even hereditary for the fake equality $=_\tau$ and is a nice construction framework, see 2.8(5).
4) $\mathfrak{u}^{[*]}$ is a nice construction framework, and if $\mathfrak{u}$ is weakly hereditary [for the fake equality $=_*$] <u>then</u> $\mathfrak{u}^{[*]}$ is hereditary [for the fake equality $=_*$].
5) If $\mathfrak{u}$ is hereditarily (for the fake equality $=_*$) <u>then</u> $\mathfrak{u}^{[*]}$ is hereditarily (for the fake equality $=_*$).

Proof. Check (really 3.1). $\square_{3.19}$

3.20 Claim. *Let $\xi \in \{\partial, \partial+1\}$ or just $\xi \le \partial + 1$. Assume $\mathfrak{u} = \text{dual}(\mathfrak{u})$ has fake equality $=_*$ and is hereditary for $=_*$.*

If $(M, N, \mathbf{I})$ has the semi ξ-uq-invariant coding property, <u>then</u> $(M, N, \mathbf{I})$ has the vertical ξ-uq-invariant coding property.

Remark. So no "$(\bar{M}, \bar{\mathbf{J}}, \mathbf{f}) \in K^{\text{qt}}_{\mathfrak{u}}$" here, but applying it we use $\mathfrak{K}^{\mathfrak{u}}$-universal homogeneous M_∂.

Proof. So let $\mathbf{d}_0$ be a $\mathfrak{u}$-free $(\alpha_{\mathbf{d}}, 0)$-rectangle satisfying $M \le_{\mathfrak{u}} M^{\mathbf{d}_0}_{0,\delta}$ and $M^{\mathbf{d}_0}_{\alpha(\mathbf{d}_0),0} \cap N = M$ so $\alpha_{\mathbf{d}_0} < \partial$ and we should find $\mathbf{d}_1$ satisfying the demand in Definition 3.10(1), this suffice. As we are assuming that "$(M, N, \mathbf{I})$ has the semi uq-invariant coding property", there is $\mathbf{e}_0$ satisfying the demands on $\mathbf{d}$ in 3.14(1)$\circledast$ with $(M, N, \mathbf{I}, M^{\mathbf{d}_0}_{0,0})$ here standing for $(M, N, \mathbf{I}, M')$ there.

Without loss of generality

$(*)_1$ $M^{\mathbf{e}_0}_{\alpha(\mathbf{e}_0),0} \cap M^{\mathbf{d}_0}_{\alpha(\mathbf{d}),0} = M^{\mathbf{d}_0}_{0,0}$.

Let

$(*)_2$ $\mathbf{e}_1 = \text{dual}(\mathbf{e}_0)$ so as $\mathfrak{u} = \text{dual}(\mathfrak{u})$ clearly $\mathbf{e}_1$ is a $\mathfrak{u}$-free $(0, \alpha(\mathbf{e}_0))$-rectangle, so $\beta(\mathbf{e}_1) = \alpha(\mathbf{e}_0)$.

Now by 1.5(5) for some $\mathbf{e}_2$ (note: even the case $\beta(\mathbf{e}_1) = \alpha(\mathbf{e}_0) = \partial$ is O.K.)

$(*)_3$ (a) $\mathbf{e}_2$ is a $\mathfrak{u}$-free $(\alpha(\mathbf{d}_0), \beta(\mathbf{e}_1))$-rectangle

(b) $\mathbf{e}_2 \restriction (\alpha(\mathbf{d}_0), 0) = \mathbf{d}_0$

(c) $\mathbf{e}_2 \restriction (0, \beta(\mathbf{e}_1))) = \mathbf{e}_1$

and without loss of generality

(d) $M^{\mathbf{e}_2}_{\alpha(\mathbf{e}_2),\beta(\mathbf{e}_2)} \cap N = M$.

Now we choose $\mathbf{d}$ by

$(*)_4$ $\mathbf{d}$ is the $\mathfrak{u}$-free $(\alpha(\mathbf{d}_0) + \alpha(\mathbf{e}_0), 0)$-rectangle such that:

(a) $M^{\mathbf{d}}_{\alpha,0}$ is $M^{\mathbf{d}_0}_{\alpha,0}$ if $\alpha \le \alpha(\mathbf{d}_0)$

(b) $M^{\mathbf{d}}_{\alpha,0}$ is $M^{\mathbf{e}_2}_{\alpha(\mathbf{d}_0),\alpha-\alpha(\mathbf{d}_0)}$ for $\alpha \in [\alpha(\mathbf{d}_0), \alpha(\mathbf{d}_0) + \alpha(\mathbf{e}_0)]$

(c) $\mathbf{J}^{\mathbf{d}}_{\alpha,0} = \mathbf{J}^{\mathbf{d}_0}_{\alpha,0}$ if $\alpha < \alpha(\mathbf{d}_0)$

(d) $\mathbf{J}^{\mathbf{d}}_{\alpha,0} = \mathbf{I}^{\mathbf{e}_2}_{\alpha(\mathbf{d}_0),\alpha-\alpha(\mathbf{d}_0)}$ if $\alpha = [\alpha(\mathbf{d}_0), \alpha(\mathbf{d}_0) + \alpha(\mathbf{e}_0))$.

[Why is this O.K.? Check; the point is that $\mathfrak{u} = \text{dual}(\mathfrak{u})$.]

And we choose $\mathbf{d}_{2,\gamma}$ for $\gamma < \min\{\alpha(\mathbf{d}_0)+1, \partial\}$

$(*)_5$ $\mathbf{d}_{2,\gamma}$ is the $\mathfrak{u}$-free $(\gamma+1,0)$-rectangle such that:

- (a) $M^{\mathbf{d}_{2,\gamma}}_{\alpha,0}$ is $M^{\mathbf{d}_0}_{\alpha,0}$ if $\alpha \le \gamma$
- (b) $M^{\mathbf{d}_{2,\gamma}}_{\alpha,0}$ is $M^{\mathbf{e}_2}_{\alpha(\mathbf{d}),\alpha(\mathbf{e}_0)}$ if $\alpha = \gamma+1$
- (c) $\mathbf{J}^{\mathbf{d}_{2,\gamma}}_{\alpha,0} = \mathbf{J}^{\mathbf{d}_0}_{\alpha,0}$ if $\alpha < \gamma$
- (d) $\mathbf{J}^{\mathbf{d}_{2,\gamma}}_{\alpha,0}$ is $\emptyset$ if $\alpha = \gamma$.

[Why is this O.K.? Check.]

So it is enough to show

$(*)_6$ $\mathbf{d}$ is as required on $\mathbf{d}$ in 3.10 for our given $\mathbf{d}_0$ and $(M, N, \mathbf{I})$.

But first note that

$(*)_7$ $\mathbf{d} = \mathbf{d}_1$ is as required in clauses (b),(c),(d) of Definition 3.10(1).

Now, modulo $(*)_7$, clearly $(*)_6$ means that we have to show that

$\boxtimes$ if $(M, N, \mathbf{I}) \le_1 (M^{\mathbf{d}}_{0,0}, N', \mathbf{I}')$ and $M^{\mathbf{d}_1}_{\alpha(\mathbf{d}_1),0} \cap N = M$, i.e. $N', \mathbf{I}'$ are as in $\otimes(e)$ of 3.10(1), <u>then</u> we shall find $\alpha', \alpha_*, \mathbf{d}_*, \mathbf{I}'', M''$ as required in clauses $(\alpha) - (\varepsilon)$ of (e) from Definition 3.10(1).

By the choice of $\mathbf{e}_0$ to be as in Definition 3.14 before $(*)_1$, for $(N', \mathbf{J}')$ from $\boxtimes$ there are $\alpha', \alpha_*, \mathbf{e}_*, N'', \mathbf{I}''_*$ such that

$\odot_1$ (α) $\alpha' \le \alpha(\mathbf{e}_0)$ and $\alpha' < \partial$

- (β) $\mathbf{e}_*$ is a $\mathfrak{u}$-free $(\alpha_*, 0)$-rectangle
- (γ) $M^{\mathbf{e}_*}_{0,0} = N'$
- (δ) there is a $\mathfrak{u}$-free $(\alpha', 1)$-rectangle $\mathbf{e}'$ such that $\mathbf{e}' \restriction (\alpha', 0) = \mathbf{e}_0 \restriction (\alpha', 0)$ and $\mathbf{e}' \restriction (\alpha', [1,1]) = \mathbf{e}_* \restriction (\alpha', 0)$
- (ε) there are no $\mathbf{d}_1, \mathbf{d}_2$ such that
  - $\bullet_1$ $\mathbf{d}_\ell$ is a $\mathfrak{u}$-free rectangle for $\ell = 1, 2$
  - $\bullet_2$ $\alpha(\mathbf{d}_1) = \alpha_*, \mathbf{d}_2 \restriction (\alpha_*, 0) = \mathbf{e}_*$
  - $\bullet_3$ $\alpha(\mathbf{d}_2, 0) \ge \alpha'$ and $\mathbf{d}_2 \restriction (\alpha', 0) = \mathbf{d} \restriction (\alpha', 0)$
  - $\bullet_4$ $(M^{\mathbf{d}_2}_{0,0}, M^{\mathbf{d}_2}_{0,1}, \mathbf{I}^{\mathbf{d}_2}_{0,0})$ is $(M', N', \mathbf{I})$ or just $\le_1$-above it
  - $\bullet_5$ there are N^*_1, N^*_2, f such that $M^{\mathbf{d}_\ell}_{\alpha(\mathbf{d}_\ell),\beta(\mathbf{d}_\ell)} \le_{\mathfrak{u}} N^*_\ell$ for $\ell = 1, 2$ and f is an isomorphism from N^*_1 onto N^*_2 over $M^{\mathbf{d}}_{\alpha(\mathbf{d}),0}$ which maps $M^{\mathbf{d}_1}_{0,\beta(\mathbf{d}_1)}$ onto $M^{\mathbf{d}_2}_{0,\beta(\mathbf{d}_2)}$.

Without loss of generality

$\odot_2$ $N'' \cap M^{\mathbf{e}_2}_{\alpha',\beta(\mathbf{e}_2)} = M^{\mathbf{e}_0}_{\alpha',0}$.

Now by induction on $\alpha \le \alpha(\mathbf{d})$ we choose $(M^*_\alpha, N^*_\alpha, \mathbf{I}^*_\alpha), \mathbf{J}^*_\alpha$ such that

$\odot_3$ (a) $(M^*_\alpha, N^*_\alpha, \mathbf{I}^*_\alpha) \in \mathrm{FR}^1_{\mathfrak{u}}$
(b) $\langle (M^*_\gamma, N^*_\gamma, \mathbf{I}^*_\gamma) : \gamma \le \alpha \rangle$ is $\le^1_{\mathfrak{u}}$-increasing continuous
(c) $(M^*_\alpha, N^*_\alpha, \mathbf{I}^*_\alpha) = (M^{\mathbf{e}_0}_{\alpha',0}, N'', \mathbf{I}'')$ for $\alpha = 0$
(d) $M^*_\alpha = M^{\mathbf{e}_2}_{\alpha,\alpha'}$
(e) $N^*_\alpha \cap M^{\mathbf{e}_2}_{\alpha(\mathbf{e}_2),\alpha'} = M^*_\alpha$
(f) if $\alpha = \alpha_1 + 1$ then $(M^{\mathbf{e}_2}_{\alpha_1,\alpha'}, M^{\mathbf{e}_2}_{\alpha_1+1,\alpha'}, \mathbf{J}^{\mathbf{e}_2}_{\alpha_1,\alpha(\mathbf{e}_0)}) \le_2 (N^*_{\alpha_1}, N^*_{\alpha_1+1}, \mathbf{J}^*_{\alpha_1})$.

Now we shall use the assumption "$\mathfrak{u}$ is hereditary for $=_*$" to finish.
Choose

$\odot_4$ f is an isomorphism from $M^*_{\alpha(\mathbf{d})}$ onto a model M^* such that $M^{\mathbf{e}_2}_{\alpha',\beta(\mathbf{e}_2)} = M^* \cap M^*_{\alpha(\mathbf{d})}$ and $f \restriction M^{\mathbf{e}_2}_{\alpha',\beta(\mathbf{e}_2)}$ is the identity as well as $f \restriction N''$

$\odot_5$ M^{**} is the unique model $\in K_{\mathfrak{u}}$ such that $M^*_{\alpha(\mathbf{d})} \subseteq M^{**}$ and $M^* \subseteq M^{**}$ and $c \in M^*_{\alpha(\mathbf{d})} \Rightarrow M^{**} \models$ "$c = f(c)$".

Lastly

$\odot_6$ $\mathbf{d}_*$ is the following $\mathfrak{u}$-free $(\alpha(\mathbf{d}) + 1, 1)$-rectangle:
(a) $\mathbf{d}_* \restriction (\alpha(\mathbf{d}), 0) = \mathbf{d}$
(b) $M^{\mathbf{d}_*}_{\alpha(\mathbf{d})+1} = M^{\mathbf{e}_2}_{\alpha(\mathbf{e}_2),\alpha}$ and $\mathbf{J}^{\mathbf{d}_*}_{\alpha(\mathbf{d}),0} = \emptyset$
(c) $M^{\mathbf{d}_*}_{\alpha(\mathbf{d})+1,1}$ is M^{**}
(d) if $\alpha \le \alpha(\mathbf{d})$ the $M^{\mathbf{d}_4}_{\alpha,1}$ is $M^{\mathbf{d}}_{\alpha,0} \cup f(M^{\mathbf{e}_2}_{\alpha,\alpha'})$, i.e. the submodel of M^{**} with this universe
(e) if $\alpha \le \alpha(\mathbf{d})$ then $\mathbf{I}^{\mathbf{d}_*}_{\alpha,0} = f(\mathbf{I}^*_\alpha)$ and $\mathbf{I}^{\mathbf{d}_*}_{\alpha(\mathbf{d})} = f(\mathbf{I}^*_{\alpha(\mathbf{d})})$
(f) if $\alpha < \alpha(\mathbf{d})$ then $\mathbf{J}^{\mathbf{d}_*}_{\alpha,1} = f(\mathbf{J}^{\mathbf{e}_2}_{\alpha,\alpha'})$.

$\square_{3.20}$

To phrase a relative of 3.20, we need:

3.21 Definition. 1) We say $\mathfrak{u}$ satisfies $(\mathrm{E})_\ell(\mathrm{f})$, is interpolative for ℓ or has interpolation for ℓ <u>when</u>:

if $(M_1, N_1, \mathbf{I}_1) \le^\ell_{\mathfrak{u}} (M_2, N_2, \mathbf{I}_2)$ then $(M_1, N_1, \mathbf{I}_1) \le^\ell_{\mathfrak{u}} (M_2, N_2, \mathbf{I}_1) \le^\ell_{\mathfrak{u}} (M_2, N_2, \mathbf{I}_2)$.

2) $(\mathrm{E})(\mathrm{f})$ means $(\mathrm{E})_1(\mathrm{f}) + (\mathrm{E})_2(\mathrm{f})$.

Remark. This is related to but is different from monotonicity, see 1.13(1).

3.22 Claim. *1) Assume that for $\ell \in \{1,2\}, \mathfrak{u}$ satisfies $(E)_\ell(f)$. For every $\mathfrak{u}$-free (α,β)-rectangle $\mathbf{d}$, also $\mathbf{e}$ is a $\mathfrak{u}$-free (α,β)-rectangle where $M^{\mathbf{e}}_{i,j} = M^{\mathbf{d}}_{i,j}, \ell = 1 \Rightarrow \mathbf{J}^{\mathbf{e}}_{i,j} = \mathbf{J}^{\mathbf{d}}_{i,j}, \ell = 1 \Rightarrow \mathbf{I}^{\mathbf{e}}_{i,j} = \mathbf{I}^{\mathbf{d}}_{0,j}$ and $\ell = 2 \Rightarrow \mathbf{J}^{\mathbf{e}}_{i,j} = \mathbf{J}^{\mathbf{d}}_{i,0}, \ell = 2 \Rightarrow \mathbf{I}^{\mathbf{e}}_{i,j} = \mathbf{I}^{\mathbf{d}}_{i,j}$.*
2) Similarly for $\mathfrak{u}$-free triangle.
3) If $\mathfrak{s}$ satisfies $(E)(f)$ then in part (1) we can let $\mathbf{J}^{\mathbf{e}}_{i,j} = \mathbf{J}^{\mathbf{d}}_{i,0}, \mathbf{I}^{\mathbf{e}}_{i,j} = \mathbf{I}^{\mathbf{d}}_{0,j}$.

Proof. Easy. $\square_{3.22}$

Now we can state the variant of 3.20.

3.23 Claim. *Assume $\xi \le \partial + 1$ and $\mathfrak{u} = \mathrm{dual}(\mathfrak{u})$ has fake equality $=_*$, is weakly hereditary for $=_*$ and has interpolation.*

If $(M, N, \mathbf{I}) \in K^{\mathrm{qt}}_{\mathfrak{u}}$ has the semi ξ-uq-invariant coding property <u>then</u> $(M, N, \mathbf{I}) \in K^{\mathrm{qt}}_{\mathfrak{u}}$ has the vertical ξ-uq-invariant coding property.

Proof. Similar to the proof of 3.20, except that

- (A) in $\odot_3$ we add $\mathbf{I}'' = \mathbf{I}'$, justified by monotonicity; note that not only is it a legal choice but still exemplify 3.10
- (B) in $\odot_4$ we add $\mathbf{I}^*_\alpha = \mathbf{I}^*_0 = \mathbf{I}'' = \mathbf{I}'$.

3.24 Theorem. *We have $\dot{I}(K^{\mathfrak{u},\mathfrak{h}}_{\partial^+}) \ge \mu_{\mathrm{unif}}(\partial^+, 2^\partial)$ <u>when</u>:*

- (a) $2^\partial < 2^{\partial^+}$
- (b) *some $(M, N, \mathbf{I}) \in \mathrm{FR}_2$ has the vertical ξ-uq-invariant coding property, see Definition 3.10*
- (c) *$\mathfrak{h}$ is a $\{0,2\}$-appropriate witness that for $\{0,2\}$-almost every $(\bar{M}, \bar{\mathbf{J}}, \mathbf{f}) \in K^{\mathrm{qt}}_{\mathfrak{u}}$, the model M_∂ is $K_{\mathfrak{u}}$-model homogeneous*
- (d) *$\xi = \partial$ <u>or</u> $\xi = \partial^+$ and $2^\theta = 2^{<\partial} < 2^\partial$.*

3.25 Remark. 1) We can phrase other theorems in this way.
2) So if we change (b) to semi uq-invariant by 3.20 it suffices to add, e.g.

(d) $\mathfrak{u}$ is hereditary for the faked equality $=_*$.

Proof. Easily by clause (b) we know $\mathfrak{u}$ has the vertical ξ-uq-invariant coding property. Now we apply 3.11, i.e. as in the proof of theorems 3.5 using 3.6(2) rather than 3.6(1) and immitating the proof of 3.3. $\square_{3.24}$

<u>3.26 Exercise</u>: Prove the parallel of the first sentence of the proof of 3.24 to other coding properties.

* * *

<u>3.27 Discussion</u>: We can repeat 3.13 - 3.24 with a game version. That is we replace Definition 3.14 by 3.28 and Definition 3.10 and then can immitate 3.20, 3.24 in 3.29, 3.30.

3.28 Definition. Let $\xi \le \partial + 1$.
1) We say that $\mathfrak{u}$ has the S-semi ξ-uq-invariant coding$_2$ property, [above $(\bar{M}^*, \bar{\mathbf{J}}^*, \mathbf{f}^*) \in K^{\rm qt}_{\mathfrak{u}}$] <u>when</u> $\{0,2\}$-almost every $(\bar{M}, \bar{\mathbf{J}}, \mathbf{f}) \in K^{\rm qt}_{\mathfrak{u}}$ [above $(\bar{M}^*, \bar{\mathbf{J}}^*, \mathbf{f}^*)$] has it, see below; if $S = \partial$ we may omit it.
2) We say that $(\bar{M}, \bar{\mathbf{J}}, \mathbf{f}) \in K^{\rm qt}_{\mathfrak{u}}$ has the S-semi ξ-invariant coding$_2$ property <u>when</u> we can choose $\langle \mathbf{d}_\delta : \delta \in S \rangle$ such that

$\circledast$ (a) $\mathbf{d}_\delta$ is a $\mathfrak{u}$-free $(\alpha(\mathbf{d}_\delta), 0)$-rectangle

(b) $M^{\mathbf{d}_\delta}_{0,0} = M_\delta$

(c) $M^{\mathbf{d}_\delta}_{\alpha(\mathbf{d}_\delta),0} \le_{\mathfrak{u}} M_\partial$

(d) in the following game the player Coder has a winning strategy; the game is defined as in Definition 2.6(2) except that the deciding who wins a play, i.e. we replace $(*)_2$ by

$(*)''_2$ in the end of the play the player Coder wins the play <u>when</u>: for a club of $\delta \in \partial$ if $\delta \in S$ <u>then</u> there are $N'', \mathbf{I}''$ such that $(M_\delta, N_\delta, \mathbf{I}_\alpha) \le_{\mathfrak{u}} (M^{\mathbf{d}_\delta}_{\alpha(\mathbf{d}_\delta),0}, N'', \mathbf{I}'')$ and for no $(\mathbf{e}, f, N_*)$ do we have the parallel of $(\alpha) - (\eta)$ of clause (c) of Definition 3.14.

3) We define when $\mathfrak{u}$ or $(\bar{M}, \bar{\mathbf{J}}, \mathbf{f})$ has the S-vertical ξ-uq-invariant coding$_2$ property as in parts (1),(2) replacing 3.14 by 3.10.

3.29 Claim. *Like 3.20 using Definition 3.28.*

3.30 Theorem. *Like 3.24 using 3.28.*

* * *

We now point out some variants of the construction framework, here amalgamation may fail (unlike 1.3(1) used in 2.10(4) but not usually). This relates to semi a.e.c.

3.31 Definition. We define when $\mathfrak{u}$ is a weak nice construction framework as in Definition 1.2 but we considerably weaken the demands of $\mathfrak{K}_{\mathfrak{u}}$ being an a.e.c.

- (A) $\mathfrak{u}$ consists of $\partial, \tau_{\mathfrak{u}}, \mathfrak{K}_{\mathfrak{u}} = (K_{\mathfrak{u}}, \le_{\mathfrak{u}})$, FR_1, $\mathrm{FR}_2, \le_1, \le_2$ (also denoted by $\mathrm{FR}^{\mathfrak{u}}_1$, $\mathrm{FR}^{\mathfrak{u}}_2, \le^1_{\mathfrak{u}}, \le^2_{\mathfrak{u}}$)
- (B) ∂ is regular uncountable
- (C) (a) $\tau_{\mathfrak{u}}$ is a vocabulary
  - (b) $K_{\mathfrak{u}}$ is a non-empty class of $\tau_{\mathfrak{u}}$-models of cardinality $< \partial$ closed under isomorphisms (but $K^{\mathfrak{u},*}_{\partial}, K^{\mathfrak{u},*}_{\partial^+}$ are defined below)
  - (c) (restricted union) if $\ell \in \{1,2\}$ then
    - (α) $\le_{\mathfrak{u}}$ is a partial order on $K_{\mathfrak{u}}$,
    - (β) $\le_{\mathfrak{u}}$ is closed under isomorphism
    - (γ) restricted union existence: if $\ell = 1,2$ and $\langle M_\alpha : \alpha \le \delta\rangle$ is $\le_{\mathfrak{K}}$-increasing continuous, δ a limit ordinal $< \partial$ and $(M_\alpha, M_{\alpha+1}, \mathbf{I}_\alpha) \in \mathrm{FR}^\ell_{\mathfrak{u}}$ for $\alpha < \delta$ and $\delta = \sup\{\alpha < \delta$: $(M_\alpha, M_{\alpha+1}, \mathbf{I}_\alpha) \in \mathrm{FR}^{\ell,+}_{\mathfrak{u}}$ <u>then</u> $M_\delta := \{M_\alpha : \alpha < \delta\}$ belongs to $K_{\mathfrak{u}}$ and $\alpha < \delta \Rightarrow M_\alpha \le_{\mathfrak{u}} M_\delta$
  - (d) restricted smoothness: in clause (c)(γ) if $\alpha < \delta \Rightarrow M_\alpha \le_{\mathfrak{u}} N$ <u>then</u> $M_\delta \le_{\mathfrak{u}} N$
- $(D)_\ell$ as in Definition 1.2
- $(E)_\ell$ as in Definition 1.2 but we replace clause (c) by
  - $(c)'$ $((M_\delta, N_\delta, \mathbf{J}_\delta) \in \mathrm{FR}^\ell_{\mathfrak{u}}$ and $\alpha < \delta \Rightarrow (M_\alpha, N_\alpha, \mathbf{J}_\alpha) \le^\ell_{\mathfrak{u}} (M_\delta, N_\delta, \mathbf{J}_\delta)$ <u>when</u>:
    - (α) $\delta < \partial$ is a limit ordinal
    - (β) $\langle (M_\alpha, N_\alpha, \mathbf{J}_\alpha) : \alpha < \delta\rangle$ is $\le^\ell_{\mathfrak{u}}$-increasing continuous
    - (γ) $(M_\delta, N_\delta, \mathbf{J}_\delta) = (\cup\{M_\alpha : \alpha < \delta\}, \cup\{N_\alpha : \alpha < \delta\}, \cup\{\mathbf{J}_\alpha : \alpha < \delta\})$
    - (δ) M_δ is a $\le_{\mathfrak{u}}$-upper bound of $\langle M_\alpha : \alpha \ge \delta\rangle$
    - (ε) N_δ is a $\le_{\mathfrak{u}}$-upper bound of $\langle N_\alpha : \alpha < \delta\rangle$
    - (ζ) $M_\delta \le_{\mathfrak{u}} N_\delta$

(F) As in Definition 1.2.

3.32 Remark. 1) We may in condition $(\mathrm{E})_\ell(\mathrm{c})$ use essentially a $\mathfrak{u}$-free $(\delta,2)$-rectangle (or $(2,\delta)$-rectangle).
2) A stronger version of $(\mathrm{E})_\ell(\mathrm{c})$ is: $(\mathrm{E})_2(\mathrm{c})^+$ as in $(\mathrm{E})_1(\mathrm{c})'$ adding:

- (η) $(M_\alpha, M_{\alpha+1}, \mathbf{I}_\alpha) \le^1_{\mathfrak{u}} (N_\alpha, N_{\alpha+1}, \mathbf{I}^1_\alpha)$
- (θ) $(M_\alpha, N_\alpha, \mathbf{I}^m_\alpha) \in \mathrm{FR}^{1,+}_{\mathfrak{u}}$ for $m = 0,1$ for unboundedly many $\alpha < \delta$ (so clause (ζ) follows).

$(\mathrm{E})_1(\mathrm{c})^+$ means clause $(\mathrm{E})_2(\mathrm{c})^+$ is satisfied by $\mathrm{dual}(\mathfrak{u})$.
3) We may demand for $(\bar M, \bar{\mathbf{J}}, \mathbf{f}) \in K^{\mathrm{qt}}_{\mathfrak{u}}$ that for a club of δ, if $\mathbf{f}(\delta) > 0$ then:

- (a) $\mathbf{f}(\delta)$ is a limit ordinal
- (b) $\mathbf{f}(\delta) = \sup\{i < \mathbf{f}(\delta) : (M_{\delta+i}, M_{\delta+i+1}, \mathbf{J}_{\delta+i}) \in \mathrm{FR}^{2,+}_{\mathfrak{u}}\}$
- (c) in the examples coming for an almost good λ-frame $\mathfrak{s}$, see §5: if $i < \delta$ and $p \in \mathscr{S}^{\mathrm{bs}}_{\mathfrak{s}}(M_{\delta+i})$ then $\delta = \sup\{j : j \in (i,\delta)$ and $\mathbf{J}_{\delta+j} = \{a_{\delta+j}\}$ and $\mathbf{tp}_{\mathfrak{s}}(a_{\delta+j}, M_{\delta+j}, M_{\delta+j+1})$ is a non-forking extension of p; see more in §5 on this.

4) We may in part (3)(b) and in 3.33(A)(a) below restrict ourselves to sucessor i.

3.33 Lemma. *We can repeat §1 + §2 (and §3) with Definition 3.31 instead of Definition 1.2 with the following changes:*

- (A) *from Definition 1.15:*
  - (a) *in the Definition of* $(\bar M, \bar{\mathbf{J}}, \mathbf{f}) \in K^{\mathrm{qt}}_{\mathfrak{u}}$ *we demand* $S = \{\delta < \partial : \mathbf{f}(\delta) > 0\}$ *is stationary and for a club of* $\delta \in S, \mathbf{f}(\delta)$ *is a limit ordinal and* $i < \mathbf{f}(\delta) \Rightarrow (M_{\delta+i}, M_{\delta+i+1}, \mathbf{J}_{\delta+i}) \in \mathrm{FR}^{2,+}_{\mathfrak{u}}$
  - (b) *we redefine* $\le^{\mathrm{qr}}_{\mathfrak{u}}, \le^{\mathrm{qs}}_{\mathfrak{u}}$ *as* $\le^{\mathrm{qt}}_{\mathfrak{u}}$ *was redefined*
- (B) *proving 1.19:*
  - (a) *in part (1), we should be given a stationary* $S \subseteq \partial$ *and for* $\alpha < \partial$ *let* $\mathbf{f}(\alpha) = \omega$
  - (b) *in part (2), we use the restricted version of union existence and smoothness*
  - (c) *in part (3), we demand* $(M^1_\alpha, M^2_\alpha, \mathbf{I}^*) \in \mathrm{FR}^+_1$ *and let* $E \subseteq \lambda\backslash\alpha$ *witness* $(\bar M, \bar{\mathbf{J}}, \mathbf{f}) \in K^{\mathrm{qt}}_{\mathfrak{u}}$ *and* $S = \{\delta \in E : \mathbf{f}(\delta) > 0\}$ *and inthe induction we use just* $\langle M_\beta : \beta \in \{\alpha\} \cup \bigcup\{[\delta,\delta,\mathbf{f}(\delta)] : \delta \in S\}\rangle$.

3.34 Exercise: Rephrase this section with τ-coding$_k$ instead coding$_k$.

[Hint: 1) Of course, we replace "coding" by "τ-coding" and isomorphic by τ-isomorphic.
2) We replace $\bullet_5$ of 3.2(1)(δ) by:

- $\bullet'_5$ there are N_1, N_2 such that $N^{\mathbf{d}_\ell}_{\alpha(\mathbf{d}_\ell),\beta(\mathbf{d}_\ell)} \leq_{\mathfrak{u}} N_\ell$ for $\ell = 1,2$ and there is a τ-isomorphism f from N_1 onto N_2 extending $\mathrm{id}_{M^{\mathbf{d}}_{\alpha',0}}$ and mapping $M^{\mathbf{d}_1}_{0,\beta(\mathbf{d}_1)}$ onto $M^{\mathbf{d}_2}_{0,\beta(\mathbf{d}_2)}$.

3) In Claim 3.3 in the end of clause (γ) "$N^1_\partial, N^2_\partial$ are not τ-isomorphic over M_γ", of course.
4) In 3.5 replace $\dot{I}$ by $\dot{I}_\tau$.
5) In 3.6, in the conclusion replace $\dot{I}$ by $\dot{I}_\tau$, in clause (c)(β) use "not τ-isomorphic".
6) In Definition 3.7 like (2), in Definition 3.10(e)(δ) as in (2).
7) Change 3.14(c)(ε) as in (2), i.e.

(ε)$'$ f is a τ-isomorphism from N^* onto N_* for some N^* such that $N'' \leq_{\mathfrak{u}} N^*$.]

§4 Straight Applications of (weak) coding

Here, to try to exemplify the usefulness of Theorem 2.3, the "lean" version, i.e. using weak coding, we revisit older non-structure results. First, recall that the aim of [Sh 603] or better VI§3,§4 is to show that the set of minimal types in $\mathscr{S}^{\mathrm{na}}_{\mathfrak{K}}(M), M \in \mathfrak{K}_\lambda$ is dense, <u>when</u>:

$\boxtimes^\lambda_{\mathfrak{K}}$ $2^\lambda < 2^{\lambda^+} < 2^{\lambda^{++}}, \mathfrak{K}$ is an a.e.c. with $\mathrm{LS}(\mathfrak{K}) \leq \lambda$, categorical in λ, λ^+ and have a medium number of models in λ^{++} (hence $\mathfrak{K}$ has amalgamation in λ and in λ^+).

More specifically we have to justify Claim VI. E46-3c.34 when the weak diamond ideal on λ^+ is not λ^{++}-saturated and we have to justify claim VI. E46-4d.17 when some $M \in K_{\lambda^+}$ is saturated; in both cases inside the proof there we quote results from here.

We interpret medium as $1 \leq \dot{I}(\lambda^{++}, K) < \mu_{\mathrm{unif}}(\lambda^{++}, 2^{\lambda^+})$ (where the latter is usually $2^{\lambda^{++}}$). This is done in 4.1-4.15, i.e., where we prove the non-structure parts relying on the one hand on the pure model theoretic part done in Chapter VI and on the other hand on coding theorems from §2. More elaborately, as we

are relying on Theorem 2.3, in 4.1 - 4.9, i.e. §4(A) we assume that the normal ideal WDmId(λ^+) is not λ^{++}-saturated and prove for appropriate $\mathfrak{u}$ that (it is a nice construction framework and) it has the weak coding property. Then in 4.10 - 4.15, i.e. §4(B) relying on Theorem 2.11, we assume more model theory and (for the appropriate $\mathfrak{u}$) prove the vertical coding property, hence eliminate the extra set theoretic assumption (but retaining the relevant cases of the WGCH, i.e. $2^\lambda < 2^{\lambda^+} < 2^{\lambda^{++}}$).

Second, we relook at the results in VI§6, i.e. [Sh 576, §6] which were originally proved relying on [Sh 576, §3]. That is, our aim is to prove the density of uniqueness triples (M,N,a) in $K^{3,\mathrm{na}}_\lambda$, assuming medium number of models in λ^{++}, and set theoretically $2^\lambda < 2^{\lambda^+} < 2^{\lambda^{++}}$ and in addition assume (for now) the non-λ^{++}-saturation of the weak diamond ideal on λ^+. So we use the "weak coding" from Definition 2.2, Theorem 2.3 (see 4.20, i.e. §4(D)). The elimination of the extra assumption is delayed as it is more involved (similarly for §4(E)).

Third, we fulfill the ("lean" version of the) promise from III§5, proving density of uniqueness triples in $K^{3,\mathrm{bs}}_{\mathfrak{s}}$, for $\mathfrak{s}$ a good λ-frame, also originally relying on [Sh 576, §3], see 4.28, i.e. §4(E).

Fourth, we deal with the promises from I§5 by Theorem 2.3 in 4.16 - 4.19, i.e. §4(C).

But still we owe the "full version", this is §4(F) in which we eliminate the extra set theoretic result relying on the model theory from §5-§8.

* * *

<u>(A) Density of the minimal types for $\mathfrak{K}_\lambda$</u>

4.1 Theorem. *We have $\dot{I}(\lambda^{++},K) \geq \mu_{\mathrm{unif}}(\lambda^{++},2^{\lambda^+})$ <u>when</u>:*

$\odot$ (a) $2^\lambda < 2^{\lambda^+} < 2^{\lambda^{++}}$

(b) *the ideal* WDmId(λ^+), *a normal ideal on λ^+, is not λ^{++}-saturated*

(c) *$\mathfrak{K}$, an a.e.c. with* LS($\mathfrak{K}$) $\leq \lambda$, *has amalgamation in λ, the* JEP *in λ, for simplicity and $K_{\lambda^+} \neq \emptyset$;*

(d) *for every $M \in K_{\lambda^+}$ and $\leq_{\mathfrak{K}}$-representation $\bar{M} = \langle M_\alpha : \alpha < \lambda^+\rangle$ of M we can find (α_0, N_0, a), i.e., a triple (M_{α_0}, N_0, a) such that:*

(α) $M_{\alpha_0} \leq_{\mathfrak{K}_\lambda} N_0$

(β) *$a \in N_0 \backslash M_{\alpha_0}$ and $\mathbf{tp}_{\mathfrak{K}}(a, M_{\alpha_0}, N_0)$ is not realized in M*

(γ) *if $\alpha_0 < \alpha_1 < \lambda^+, M_{\alpha_1} \leq_{\mathfrak{K}} N_1$ and f is a $\leq_{\mathfrak{K}}$-embedding of N_0 into N_1 over M_{α_0} <u>then</u> we can find $\alpha_2 \in (\alpha_1,\lambda^+)$ such that M_{α_2}, N_1 are not uniquely amalgamated over M_{α_1} (in $\mathfrak{K}_\lambda$), i.e.* $\mathrm{NUQ}_\lambda(M_{\alpha_1}, M_{\alpha_2}, N_1)$, *see VI. E46-3c.4 (2).*

4.2 Remark. 1) Used in Claim VI.E46-3c.34, more exactly the relative 4.3 is used.
2) A further question, mentioned in VI.E46-2b.33(3) concern $\dot{I}\dot{E}(\lambda^{++}, K)$ but we do not deal with it here.
3) Recall that for $M \in K_\lambda$ the type $p \in \mathscr{S}^{\rm na}_{\mathfrak{K}_\lambda}(M)$ is minimal <u>when</u> there is no $M_2 \in K_\lambda$ which $\le_{\mathfrak{K}}$-extends M and p has at least 2 extensions in $\mathscr{S}^{\rm na}_{\mathfrak{K}_\lambda}(M_2)$; see Definition VI.E46-1a.34.
3A) We say that in $\mathfrak{K}_\lambda$ the minimal types are dense when: for any $M \in K_\lambda$ and $p \in \mathscr{S}^{\rm na}_{\mathfrak{K}_\lambda}(M)$ there is a pair (N, q) such that $M \le_{\mathfrak{K}_\lambda} N$ and $q \in \mathscr{S}^{\rm na}_{\mathfrak{K}_\lambda}(N)$ is minimal and extend p (see VI.E46-1a.34(1A)).
4) A weaker version of Clause (d) of 4.1 holds when any $M \in K_{\lambda^+}$ is saturated (above λ) and the minimal types are not dense (i.e. omit subclause (β) and in subclause (γ) add $f(a) \notin M_{\alpha_1}$; the proof is similar (but using 4.11). Actually, 4.1 as phrased is useful normally only when $2^\lambda > \lambda^+$, but otherwise we use 4.3.
5) In VI.E46-3c.34 we work more to justify a weaker version (d)$''$ of Lemma 4.3 below which suffice.

Similarly

4.3 Lemma. *1) Like 4.1 but we replace clause (d) by:*

> $(d)'$ *there is a superlimit $M \in K_{\lambda^+}$ and for it clause (d) of 4.1 holds.*

2) For τ a K-sub-vocabulary, see 1.8(5), we have $\dot{I}_\tau(\lambda^{++}, K) \ge \mu_{\rm unif}(\lambda^{++}, 2^{\lambda^+})$ when (a),(b),(c) of 4.1 holds and

> $(d)''$ *there[22] is K''_{λ^+} such that*
> > (α) *$K''_{\lambda^+} \subseteq K_{\lambda^+}, K''_{\lambda^+} \ne \emptyset$ and K''_{λ^+} is closed under unions of $\le_{\mathfrak{K}}$-increasing continuous sequences of length $\le \lambda^+$*
> >
> > (β) *there is $M_* \in K''_{\lambda^+}$ such that for any $\le_{\mathfrak{K}_\lambda}$-increasing continuous sequence $\langle M^1_\alpha : \alpha < \lambda^+\rangle$ with union $M^1 \in K''_{\lambda^+}$ satisfying $M_* \le_{\mathfrak{K}} M^1$, in the following game the even player has a winning strategy. In the α-th move a triple $(\beta_\alpha, M_\alpha, f_\alpha)$ is chosen such that $\beta_\alpha < \lambda^+, M_\alpha \in \mathfrak{K}_\lambda$ has universe $\subseteq \lambda^+$ and f_α is a $\le_{\mathfrak{K}}$-embedding of $M^1_{\beta_\alpha}$ into M_α, all three are increasing continuous with α. Of course, the α-th move is done by the even/odd iff α is even/odd. Lastly, in the end the even player*

[22] Why not K'_λ? Just because we use this notation in 1.10.

wins iff $\cup\{M_\alpha : \alpha < \lambda^+\}$ *belongs to* K''_{λ^+} *and for a club of* $\alpha < \lambda^+$ *for some* $\gamma \in (\beta_\alpha, \lambda^+)$ *and some* $N \in K_\lambda$ *such that there is an isomorphism from* N *onto* M_α *extending* f_α *we have* $\mathrm{NUQ}_\tau(M^1_{\beta_\alpha}, N, M^1_\gamma)$, *i.e.* N, M_γ *can be amalgamated over* M_{β_α} *in* $\mathfrak{K}_\lambda$ *in at least two* τ-*incompatible ways.*

4.4 Remark. 0) We may replace β_α in clause (d)$''(\beta)$ above by $\beta = \alpha + 1$, many times it does not matter.
1) We may weaken the model theoretic assumption $(d)''$ of 4.3(2) so weaken (d) in 4.1 and $(d)'$ in 4.3 if we strengthen the set theoretic assumptions, e.g.

- $(*)_1$ for some stationary $S \subseteq S^{\lambda^{++}}_{\lambda^+}$ we have $S \in \check{I}[\lambda^{++}]$ but $S \notin \mathrm{WDmId}(\lambda^{++})$
- $(*)_2$ in 4.3, in subclause (α) of clause (d)$''$ we weaken the closure under union of K''_{λ^+} to: for a $\le_{\mathfrak{K}}$-increasing sequence in K''_{λ^+} of length λ^+, its union belongs to K''_{λ^+}.

2) If, e.g. $\lambda = \lambda^{<\lambda}$ and $\mathbf{V} = \mathbf{V}^{\mathbb{Q}}$ where $\mathbb{Q}$ is the forcing notion of adding λ^+-Cohen subsets of λ and the minimal types are not dense <u>then</u> $\dot{I}(\lambda^+, K) = 2^{\lambda^+}$, (hopefully see more in [Sh:E45]).
3) If in part (2) of 4.3, we may consider omitting the amalgamation, but demand "no maximal model in $\mathfrak{K}_\lambda$". However, the minimality may hold for uninteresting reasons.
4) This is used in VI. E46-3c.34.
5) We may assume $2^\lambda \ne \lambda^+$ as essentially the case $2^\lambda = \lambda^+$ is covered[23] by Lemma 4.10 below.
6) In clause (d)$''(\beta)$ of Lemma 4.3, we can let the even player choose also for $\alpha = \delta + 1$ for $\delta \in S$ <u>when</u> $S \subseteq \lambda^+$ but $\mathrm{WDmId}(\lambda^+) + S$ is not λ^{++}-saturated.

Proof of 4.1. We shall apply Theorem 2.3. So (model theoretically) we have an a.e.c. $\mathfrak{K}$ with $\mathrm{LS}(\mathfrak{K}) \le \lambda$, and we are interested in proving $\dot{I}(\lambda^{++}, \mathfrak{K}) \ge \mu_{\mathrm{unif}}(\lambda^{++}, 2^{\lambda^+})$.

We shall define (in Definition 4.5 below) a nice construction framework $\mathfrak{u}$ such that $\partial_{\mathfrak{u}} = \lambda^+$; the set theoretic assumptions of 2.3 hold; i.e.

[23] by the assumption $\mathfrak{K}_\lambda$ has amalgamation so togther with $2^\lambda = \lambda^+$, if $|\tau_K| \le \lambda$ or just $M \in \mathfrak{K}_\lambda \Rightarrow |\mathscr{S}_{\mathfrak{K}_\lambda}(M)| \le \lambda^+$ <u>then</u> there is a model $M^* \in \mathfrak{K}_{\lambda^+}$ which is saturated above λ; in general this is how it is used in VI§4; but if there is $M \in \mathfrak{K}_\lambda$ with $|\mathscr{S}_{\mathfrak{K}_\lambda}(M)| > \lambda^+$; we can use 4.1.

(a) $\lambda < \partial$ and $2^\lambda = 2^{<\partial} < 2^\partial$; i.e. we choose $\theta := \lambda$ and this holds by clause $\odot(a)$ of Theorem 4.1

(b) $2^\partial < 2^{\partial^+}$; holds by clause $\odot(a)$ of 4.1

(c) the ideal WDmID(∂) is not ∂^+-saturated; holds by clause $\odot(b)$ of 4.1

We still have to find $\mathfrak{u}$ (and τ) as required in clause (d) of Theorem 2.3. We define it in Definition 4.5 below, in particular we let $\mathfrak{K}_{\mathfrak{u}} = \mathfrak{K}'_\lambda, \tau = \tau_{\mathfrak{K}}$, see Definition 1.10 where $\mathfrak{K}$ is the a.e.c. from 4.3 hence the conclusion "$\dot{I}_\tau(\partial^+, K^{\mathfrak{u},\mathfrak{h}}_{\partial^+}) \geq \mu_{\rm unif}(\partial^+, 2^\partial) > 2^\partial$ for any $\{0,2\}$-appropriate function $\mathfrak{h}$" of Theorem 2.3 implies that $\dot{I}(\lambda^{++}, K^{\mathfrak{u}}) \geq \mu_{\rm unif}(\lambda^{++}, 2^{\lambda^+})$ as required using 4.7(3); we can use Exercise 1.12. So what we should actually prove is that we can find such nice construction frameworks $\mathfrak{u}$ with the weak coding property which follows from $\mathfrak{u}$ having the weak coding property (by 2.10(2),(3)) and $\mathfrak{h}$ such that every $M \in K^{\mathfrak{u},\mathfrak{h}}_{\lambda^{++}}$ is τ-fuller. This is done in 4.6, 4.7 below. $\square_{4.1}$

Proof of 4.3. 1) By part (2), in particular letting $K''_{\lambda^+} = \{M \in K_{\lambda^+} : M$ is superlimit$\}$.

But why does subclause (β) of clause (d)$''$ hold? Let $M_* \in K''_{\lambda^+}$ be superlimit, let $\langle M^1_\alpha : \alpha < \lambda^+\rangle$ be $\leq_{\mathfrak{K}_\lambda}$-increasing continuous with union $M^1 \in K''_{\lambda^+}$ and assume $M_* \leq_{\mathfrak{K}} M^1$. Without loss of generality M^1 has universe $\{\beta < \lambda^+ : \beta$ odd$\}$.

We shall prove that the even player has a winning strategy. We describe it as follows: the even player in the α-th move also choose (N_α, g_α) for α even and also for α odd (after the odd's move) such that

$(*)$ (a) $N_\alpha \in K_{\lambda^+}$ is superlimit and $M^1 \leq_{\mathfrak{K}} N_0$

(b) the universe of N_α is $\{\gamma < \lambda^+ : \gamma$ is not divisible by $\lambda^\alpha\}$

(c) $N_\beta \leq_{\mathfrak{K}} N_\alpha$ for $\beta < \alpha$

(d) g_α is a $\leq_{\mathfrak{K}}$-embedding of M_α into N_α

(e) $g_\beta \subseteq g_\alpha$ for $\beta < \alpha$

(f) if α is odd then the universe of $g_{\alpha+1}(M_{\alpha+1})$ includes $\alpha = N_\alpha \cap \alpha$.

It should be clear that the even player can do this. Also for any such play $\langle (M_\alpha, f_\alpha, N_\alpha, g_\alpha) : \alpha < \lambda^+\rangle$ we have $\lambda^+ = \cup\{N_\alpha \cap \alpha : \alpha < \lambda^+\} \subseteq \cup\{g_\alpha(M_\alpha) : \alpha < \lambda^+\} \subseteq \cup\{N_\alpha : \alpha < \lambda^+\} \subseteq \lambda^+$, so $g = \cup\{g_\alpha : \alpha < \lambda^+\}$ is an isomorphism from $\cup\{M_\alpha : \alpha < \lambda^+\}$ onto $\cup\{N_\alpha : \alpha < \lambda^+\}$. As the latter is superlimit we are done (so for this part being (λ^+, λ^+)-superlimit suffice).

2) The proof is like the proof of 4.1 but we have to use a variant of 2.3, i.e. we use the variant of weak coding where we use a game, see 2.5. $\square_{4.3}$

4.5 Definition. [Assume clause (c) of 4.1.]

We define $\mathfrak{u} = \mathfrak{u}^1_{\mathfrak{K}_\lambda}$ as follows (with $\tau(\mathfrak{u}) = (\tau_{\mathfrak{K}})'$ so $=_\tau$ is a congruence relation in $\tau(\mathfrak{u})$, O.K. by 1.10; this is a fake equality 3.17(1), 3.19)

- (a) $\partial_{\mathfrak{u}} = \lambda^+$
- (b) essentially $\mathfrak{K}_{\mathfrak{u}} = \mathfrak{K}_\lambda$; really $\mathfrak{K}'_\lambda$ (i.e. $=_\tau$ is a congruence relation!)
- (c) $\mathrm{FR}_1 = \{(M, N, \mathbf{I}) : M \le_{\mathfrak{K}_{\mathfrak{u}}} N, \mathbf{I} \subseteq N \backslash M$ empty or a singleton $\{a\}\}$
- (d) $\mathrm{FR}_2 =\ \mathrm{FR}_1$
- (e) $(M_1, N_1, \mathbf{J}_1) \le_\ell (M_2, N_2, \mathbf{J}_2)$ <u>when</u>
  - (i) both triples are from FR_ℓ
  - (ii) $M_1 \le_{\mathfrak{K}} M_2, N_1 \le_{\mathfrak{K}} N_2$ and $\mathbf{J}_1 \subseteq \mathbf{J}_2$
  - (iii) $M_2 \cap N_1 = M_1$.

4.6 Observation. $\mathfrak{u}$ is a nice construction framework which is self-dual.

Proof. Easy and the proof of 4.13 can serve when we note that $(\mathrm{D})_1(\mathrm{d})$ and (F) are obvious in our context, recalling we have fake equality. $\square_{4.6}$

4.7 Observation. [Assume $\lambda, \mathfrak{K}$ are as in 4.1 or 4.3(1) or 4.3(2).]

1) $\mathfrak{u}$ has the weak coding property (see Definition 2.2).

2) If $(\bar M, \bar{\mathbf{J}}, \mathbf{f}) \in K^{\mathrm{qt}}_{\mathfrak{u}}$ and so $M_\partial \in K_{\lambda^+}$ but for 4.3(2) the model M_∂ is superlimit <u>then</u> $(\bar M, \bar{\mathbf{J}}, \mathbf{f})$ has weak coding property.

3) For some $\mathfrak{u} - 0$-appropriate function $\mathfrak{h}$, every $M \in K^{\mathfrak{u},\mathfrak{h}}_{\lambda^{++}}$ is $\tau_{\mathfrak{s}}$-fuller, i.e. the model $M/=^M$ has cardinality λ^{++} and the set $a/=^M$ has cardinality λ^{++} for every $a \in M$.

Proof. Easy.

1) By part (2) and (3).

2) For proving 4.1 by clause (d) there we choose $(\alpha(0), N_0, \mathbf{I}_0)$, it is as required in Definition 2.2(3), noting that we can get the necessary disjointness because $\mathfrak{K}'_\lambda$ has fake equality. Similarly for 4.3(1).

For proving 4.3(2) we fix a winning strategy $\mathbf{st}$ for the even player in the game from clause (d) of 4.3. Again by the fake equality during the game we can demand $\alpha_1 < \alpha \Rightarrow f_\alpha(M_{\beta_\alpha}) \cap M_\alpha = f_{\alpha_1}(M_{\beta_1})$.

3) By 1.25(3) is suffice to deal separately with each aspect of being $\tau_{\mathfrak{s}}$-fuller.

First, we choose a $\mathfrak{u}$-0-appropriate function $\mathfrak{h}_0$ such that if $((\bar M^1, \bar{\mathbf{J}}^1, \mathbf{f}^1), (\bar M^2, \bar{\mathbf{J}}^2, \mathbf{f}^2))$ does 0-obeys $\mathfrak{h}_0$ as witnessed by $(E, \bar{\mathbf{I}})$ then for any $\delta \in E, (M^1_\delta, M^2_\delta, \mathbf{I}_\delta) \in \mathrm{FR}^+_1$ and there is $a \in \mathbf{I}_\delta$ such that $c \in M^1_\partial \Rightarrow M^2_\partial \models \neg(a =_\tau c)$; this is possible as in every $(\bar M^1, \bar{\mathbf{J}}^1, \mathbf{f}^1) \in K^{\mathrm{qt}}_{\mathfrak{u}}$ there are $\alpha < \lambda^+$ and $p \in \mathscr{S}^{\mathrm{na}}(M^1_\alpha)$ not realized in $M^1 = \cup\{M^1_\beta : \beta < \alpha\}$. Why? For 4.1 by 4.1(d)(β), similarly for 4.3(1) and for 4.3(2), if it fails, then the even player cannot win, because

- $(*)$ if $M_0 \le_{\mathfrak{u}} M_\ell$ for $\ell = 1,2$ and $(\forall b \in M_1)(\exists b \in M_1)(\exists a \in M_0)(M_1 \models a =_\tau b)$ then M_1, M_2 can be uniquely disjointly amalgamated in $\mathfrak{K}_{\mathfrak{u}}$.

Second, we choose a $\mathfrak{u} - 0$-suitable $\mathfrak{h}_2$ such that if $\langle (\bar M^\alpha, \bar{\mathbf{J}}^\alpha, \mathbf{f}^\alpha) : \alpha < \partial^+\rangle$ does 0-obeys $\mathfrak{h}_2$, then for every $\alpha < \partial^+$ and $a \in M_\alpha$ for λ^{++} many $\delta \in (\alpha, \lambda^{++})$, we have $(a/ =^{M_\delta}_\tau) \subset (a/ =^{M_{\delta+1}}_\tau)$. $\square_{4.7}$

4.8 Example For $\mathfrak{K}, \mathfrak{u}$, $\mathbf{st}$ as in the proof of 4.7(2).
1) For some initial segment $\mathbf{x} = \langle (\beta_\alpha, M_\alpha, f_\alpha) : \alpha \le \alpha_*\rangle$, of a play of the game of length $\alpha_* < \lambda^+$ in which the even player uses the strategy $\mathbf{st}$, for any longer such initial segment $\langle (\beta_\alpha, M_\alpha, f_\alpha) : \alpha \le \alpha_{**}\rangle$ of such a play we have $M_{\alpha_*} \cap f_{\alpha_{**}}(M^1_{\beta_{\alpha_{**}}}) = f_{\alpha_*}(M^1_{\beta_{\alpha_*}})$ and $f_{\alpha_*}(M^1_{\beta_{\alpha_*}}) <_{\mathfrak{K}} M_\alpha$.
[Why? As in the proof of the density of reduced triples; just think.]
2) Moreover if $c \in M_\partial \backslash M_{\beta_{\alpha_*}}$ then $f_{\alpha_*}(\mathbf{tp}_{\mathfrak{K}}(c, M_{\beta_{\alpha_*}}, M_\partial))$ is not realized in M_{α_*} (recall the definition of NUQ). $\square_{4.1}$

4.9 Remark. So we have finished proving 4.1, 4.3.

* * *

(B) Density of minimal types: without λ^{++}-saturation of the ideal

The following takes care of VI. E46-4d.17, of its assumptions, (a)-(g) are listed in VI. E46-4d.14.

4.10 Theorem. *We have $\dot I(\lambda^{++}, K) \ge \mu_{\mathrm{unif}}(\lambda^{++}, 2^{\lambda^+})$ when:*

- (a) $2^\lambda < 2^{\lambda^+} < 2^{\lambda^{+2}}$
- (b) *$\mathfrak{K}$ is an abstract elementary class,* $\mathrm{LS}(\mathfrak{K}) \le \lambda$
- (c) $K_{\lambda^{++}} \neq \emptyset$,
- (d) *$\mathfrak{K}$ has amalgamation in λ*
- (e) *the minimal types, for $\mathfrak{K}_\lambda$ are not dense, see 4.2(3A)*
- (f) *$\mathfrak{K}$ is categorical in λ^+ or at least has a superlimit model in λ^+*
- (g) *there is $M \in K_{\lambda^+}$ which is saturated (in $\mathfrak{K}$) above λ.*

Proof. The proof is broken as in the other cases.

4.11 Definition. We define $\mathfrak{u} = \mathfrak{u}^2_{\mathfrak{K}_\lambda}$ as in 4.5 so $\mathfrak{K}_{\mathfrak{u}} = \mathfrak{K}'_\lambda$ but replacing clauses (c),(e) by (we shall use the fake equality only for having disjoint amalgamation):

- $(c)'$ $\mathrm{FR}_1 = \{(M,N,\mathbf{I}) : M \le_{\mathfrak{K}_{\mathfrak{u}}} N, \mathbf{I} \subseteq N\backslash M$ empty or a singleton $\{a\}$ and if $(a/=^M) \notin M/=^M$ <u>then</u> $\mathbf{tp}_{\mathfrak{K}_\lambda}(a,M,N)$ has no minimal extension, i.e. and $\mathbf{tp}_{\mathfrak{K}_\lambda}((a/=^M_\tau,(M/=^M_\tau),(N/=^N_\tau))$ has no minimal extension$\}$
- (e) $(M_1,N_1,\mathbf{J}_1) \le_\ell (M_2,N_2,\mathbf{J}_2)$ <u>when</u>: clauses (i),(ii),(iii) there and
  - (iv) if $b \in \mathbf{J}_1$ and $(\forall a \in M_1)(\neg a =^{N_1}_\tau b)$ <u>then</u> $(\forall a \in M_2)(\neg a =^{N_2}_\tau b)$.

4.12 Observation. Without loss of generality $\mathfrak{K}$ has $(\mathrm{jep})_\lambda$ and for every $M \in K_\lambda$ there is $p \in \mathscr{S}^{\mathrm{na}}_{\mathfrak{K}_\lambda}(M)$ with no minimal extension.

Proof. Why?

Because if (p_*,M_*) are as required we can replace $\mathfrak{K}$ by $\mathfrak{K}^* = \mathfrak{K} \restriction \{M \in K$: there is a $\le_{\mathfrak{K}}$-embedding of M_* into $M\}$. Clearly $\mathfrak{K}^*$ satisfies the older requirements and if h is a $\le_{\mathfrak{K}}$-embedding of M_* into $M \in \mathfrak{K}^*_\lambda$ then $h(p_*)$ can be extended to some $p \in \mathscr{S}_{\mathfrak{K}^*_\lambda}(M) = \mathscr{S}_{\mathfrak{K}_\lambda}(M)$ as required. Why it can be extended? As any triples $(M,N,a) \in K^{3,\mathrm{na}}_\lambda$ with no minimal extension has the extension property, see VI. E46-2b.7(1). $\square_{4.12}$

The first step is to prove that (and its proof includes a proof of 4.6).

4.13 Claim. $\mathfrak{u}$ *is a nice construction framework (if it is as in 4.12).*

Proof. Clauses (A),(B),(C) of Definition 1.2 are obvious. Also $(\mathrm{D})_1 = (\mathrm{D})_2$ and $(\mathrm{E})_1 = (\mathrm{E})_2$ as $\mathrm{FR}_1 = \mathrm{FR}_1$ and $\le_1 = \le_2$. Now $(\mathrm{D})_1$(a), (b), (c), (e) and $(\mathrm{E})_1$(a), (b)(α), (c), (d) holds by the definition of $\mathfrak{u}$. Concerning $(\mathrm{D})_1$(d), by assumption (e) of Lemma 4.10 clearly $\mathrm{FR}^+_1 \neq \emptyset$ and by Observation 4.12 we have $[M \in K_\lambda \Rightarrow (M,N,a) \in \mathrm{FR}_1$ for some pair $(N,\mathbf{I})]$, i.e. $(\mathrm{D})_1$(d) holds. As $\mathfrak{K}_\lambda$ has amalgamation and $(\mathrm{D})_1$(d) holds, clearly

- $(*)$ if $M \le_{\mathfrak{K}_\lambda} N$ then for some pair $(N',\mathbf{J})$ we have $N \le_{\mathfrak{K}_\lambda} N'$ and $(M,N',\mathbf{J}) \in \mathrm{FR}^+_1$.

Concerning $(\mathrm{E})_1(\mathrm{b})(\beta)$, just remember that $=_\tau$ is a fake equality, recalling subclause (iii) of clause (e) of Definition 4.5. Now the main point is amalgamation = clause (F) of Definition 1.2. We first ignore the $=_\tau$ and disjointness, that is, we work in $\mathfrak{K}_\lambda$ not $\mathfrak{K}'_\lambda$; easily this suffices. So we assume that $(M_0, M_\ell, \mathbf{J}_\ell) \in \mathrm{FR}_\ell$ for $\ell = 1,2$ and by $(*)$ above without loss of generality $\ell = 1,2 \Rightarrow \mathbf{J}_\ell \neq \emptyset$ so let $\mathbf{J}_\ell = \{a_\ell\}$. Let $p_\ell = \mathbf{tp}_{\mathfrak{K}_\lambda}(a_\ell, M_0, M_\ell)$ and by VI. E46-1a.43(1) we can find a reduced $(M'_0, M'_1, a_1) \in K^{3,\mathrm{na}}_\lambda$ which is $\leq_{\mathrm{na}}$-above (M_0, M_1, a_1). We can apply Claim VI. E46-2b.7(1) because: first Hypothesis VI. E46-2b.0 holds (as $\mathfrak{K}$ is an a.e.c., $\mathrm{LS}(\mathfrak{K}) \leq \lambda$ and $K_\lambda \neq \emptyset$) and, second, $(\mathrm{amg})_\lambda$ holds by assumption (d) of 4.10. So by VI. E46-2b.7(1), the extension property for such types (i.e. ones with no minimal extensions) holds, so there are M'_2 such that $M'_0 \leq_{\mathfrak{K}_\lambda} M'_2$ and $\leq_{\mathfrak{K}}$-embedding g of M_2 into M'_2 over M_0 such that $g(a_2) \notin M'_0$.

Again by VI. E46-2b.7(1) we can find $M''_1 \in K_\lambda$ such that $M'_2 \leq_{\mathfrak{K}} M''_1$ and f which is a $\leq_{\mathfrak{K}}$-embedding of M'_1 into M''_1 such that $f(a_1) \notin M'_2$.

By the definition of "(M'_0, M'_1, a) is reduced", see Definition VI. E46-1a.34 it follows that $f(M'_1) \cap M'_2 = M'_0$, so $M'_1 \cap M'_2 = M'_0$. In particular $f(a_1) \in M'_2 \wedge g(a_2) \notin f(M'_1)$ so we are done. Now the result with disjointness follows because $=_\tau$ is a fake equality. $\square_{4.13}$

4.14 Claim. *1) $\mathfrak{u}$ has the vertical coding property, see Definition 2.9(5).*
2) If $(M, N, \mathbf{I}) \in \mathrm{FR}^+_2$ and $a \in \mathbf{I}$ & $b \in M \Rightarrow \neg(a =^{N_1}_\tau b)$ <u>then</u> this triple has the true vertical coding$_0$ property (see Definition 2.9(1B).
3) $\mathfrak{K}_{\lambda^+}$ has a superlimit model which is saturated.
4) For almost$_2$ every $(\bar{M}, \bar{\mathbf{J}}, \mathbf{f}) \in K^{\mathrm{qt}}_{\mathfrak{u}}$ the model M_∂ is saturated.
5) Every $(\bar{M}, \bar{\mathbf{J}}, \mathbf{f}) \in K^{\mathrm{qt}}_{\mathfrak{u}}$ has the vertical coding property (see Definition 2.9(3)) <u>when</u> $M_{\lambda^+} \in \mathfrak{K}_{\lambda^+}$ is saturated.
6) For some $\mathfrak{u} - 0$-appropriate function $\mathfrak{h}$, for every $M \in K^{\mathfrak{u},\mathfrak{h}}_{\lambda^{++}}$ the model $M/=^M$ have cardinality λ^{++} and the set $a/=^M_\tau$ has cardinality λ^{++} for every $a \in M$.

Proof. 1) Follows by part (3),(4),(5).
2) By the choice of $\mathrm{FR}^{\mathfrak{u}}_1$ for some $a, \mathbf{I} = \{a\}$ and the type $\mathbf{tp}_{\mathfrak{K}_\lambda}(a, M, N)$ has no minimal extension. To prove the true vertical coding property assume that $(\langle M^\ell_i : i \leq \beta\rangle, \langle \mathbf{J}^\ell_i : i < \beta\rangle)$ for $\ell = 1,2$ and $\langle \mathbf{I}_i : i \leq \beta\rangle$ are as in Definition 2.9(1), i.e., they form a $\mathfrak{u}$-free $(\beta,1)$-rectangle with $(M, N, \{a\}) \leq^1_{\mathfrak{u}} (M^1_0, M^2_0, \mathbf{I}_0)$; i.e. there is $\mathbf{d}$, a $\mathfrak{u}$-free $(\beta,1)$-rectangle such that $M^{\mathbf{d}}_{i,0} = M^1_i, M^{\mathbf{d}}_{i,1} = M^2_i, \mathbf{J}^{\mathbf{d}}_{i,\ell} = \mathbf{J}^\ell_i, \mathbf{I}^{\mathbf{d}}_{i,\ell} = \mathbf{I}_i$).

So $M \leq_{\mathfrak{K}_\lambda} M^1_\beta, N \leq_{\mathfrak{K}_\lambda} M^2_\beta, a \in N \backslash M^1_\beta$ and $\mathbf{I}_i = \{a\}$. As $\mathbf{tp}_{\mathfrak{K}_\lambda}(a, M, N)$ has no minimal extension we can find $M^1_{\beta+1}$ such that $M^1_\beta \leq_{\mathfrak{K}_\lambda} M^1_{\beta+1}$ and $\mathbf{tp}_{\mathfrak{K}_\lambda}(a, M^1_\beta, M^2_\beta)$ has at least two non-algebraic extensions in $\mathscr{S}_{\mathfrak{K}_\lambda}(M^1_\beta)$, hence we can choose $p_1 \neq p_2 \in \mathscr{S}^{\mathrm{na}}_{\mathfrak{K}_\lambda}(M^1_{\beta+1})$ extending $\mathbf{tp}_{\mathfrak{K}_\lambda}(a, M^1_\beta, M^2_\beta)$. Now treating equality as congruence

without loss of generality $M^1_{\beta+1} \cap M^2_\beta = M^1_\beta$ and there are $N^1, N^2 \in K_\lambda$ such that $M^1_{\beta+1} \le_{\mathfrak{K}} N^\ell, M^2_\beta \le_{\mathfrak{K}} N^\ell$ and $\mathbf{tp}_{\mathfrak{K}_\lambda}(a, M^1_{\beta+1}, N_\ell) = p_\ell$ for $\ell = 1, 2$.
Letting $M^{2,\ell}_{\beta+1} := N_\ell$ we are done.
3) If $\mathfrak{K}$ is categorical in λ^+ then the desired conclusion holds as every $M \in \mathfrak{K}_{\lambda^+}$ is saturated above λ by clause (g) of the assumption of 4.10. If $\mathfrak{K}$ only has a superlimit model in $\mathfrak{K}_{\lambda^+}$ as there is a $M' \in \mathfrak{K}_{\lambda^+}$ saturated above λ, necessarily the superlimit $M' \in \mathfrak{K}_{\lambda^+}$ is saturated above λ by VI.E46-2b.13(4).
4) We prove the existence of $\mathfrak{g}$ for the "almost$_2$" (or use the proof of 4.3(1)). Now recalling $(*)_4$ of 1.22(1) for each $M \in K_\lambda$ with universe $\in [\lambda^{++}]^\lambda$, we can choose a sequence $\langle p_{M,\alpha} : \alpha < \lambda^+\rangle$ listing $\mathscr{S}_{\mathfrak{K}_\lambda}(M)$. When defining the value $\mathfrak{g}(\bar M^1, \bar{\mathbf{J}}^1, \mathbf{f}^1, \bar M^2 \restriction (\delta + \mathbf{f}^1(\delta) + 1)), \bar{\mathbf{J}}^2 \restriction (\delta + \mathbf{f}^1(\delta), \bar{\mathbf{I}} \restriction (\delta + \mathbf{f}^1(\delta) + 1), S)$, see Definition 1.22(1)(c) we just realize all $p_{M^2_i,j}$ with $i, j < \delta$. Recalling that by part (3) the union of a $\le_{\mathfrak{K}}$-increasing sequence of length $< \lambda^{++}$ of saturated members of K_{λ^+} is saturated, we are done.
5) Holds by Observation 2.10(1).
6) Easy, as in 4.7(3). $\square_{4.14}$

Continuation of the Proof of 4.10. By the 1.19, 4.13, 4.14(1) we can apply Theorem 2.11. $\square_{4.10}$

4.15 Remark. So we have finished proving 4.10.

* * *

<u>(C) The symmetry property of $\mathrm{PC}_{\aleph_0}$ classes</u>

Here we pay a debt from Theorem I.88r-5.23(1), so naturally we assume knowledge of I§5; of course later results supercede this. Also we can avoid this subsection altogether, dealing with the derived good $\aleph_0$-frame in III.600-Ex.1.

4.16 Theorem. $\dot I(\aleph_2, \mathfrak{K}) \ge \mu_{\mathrm{unif}}(\aleph_2, 2^{\aleph_1})$ *moreover* $\dot I(\aleph_2, \mathfrak{K}(\aleph_1\text{-saturated})) \ge \mu_{\mathrm{unif}}(\aleph_2, 2^{\aleph_1})$ <u>*when:*</u>

$\circledast$ (a) *(set theory)*

(α) $2^{\aleph_0} < 2^{\aleph_1} < 2^{\aleph_0}$ *and*

(β) $\mathrm{WDmId}(\aleph_1)$ *is not* $\aleph_2$*-saturated*

(b) $\mathfrak{K}$*, an a.e.c., is* $\aleph_0$*-representable, i.e., is* $\mathrm{PC}_{\aleph_0}$*-a.e.c., see Definition I.* *88r-1.4* *(4),(5)*

(c) $\mathfrak{K}$ *is as in I.* *88r-4.5*, *I.* *88r-5.0*

(d) $\mathfrak{K}$ *fails the symmetry property or the uniqueness of two sided stable amalgamation, see Definition I.* *88r-5.21*, *equivalently*

(d)′ $\mathfrak{K}$ *fails the uniqueness of one-sided amalgamation*

(e) $\mathbf{D}$ *is countable, see Definition I.* *88r-5.1*, *I.* *88r-5.6.1* *and III§3(B).*

Remark. 1) On omitting "WDmId($\aleph_1$) is not $\aleph_2$-saturated", see Conclusion 4.35.
2) Clause (e) of 4.16 is reasonable as we can without loss of generality assume it by Observation I. 88r-5.23.8.

Proof. Let $\lambda = \aleph_0$, without loss of generality

(f) for $M \in K$, any finite sequence is coded by an element.

Now by III§3 (B), i.e. III. 600-Ex.1 we have $\mathfrak{s}_{\aleph_0}$, which we call $\mathfrak{s} = \mathfrak{s}_{\mathfrak{K}} = \mathfrak{s}^1_{\mathfrak{K}}$, is a good $\aleph_0$-frame as defined (and proved) there, moreover $\mathfrak{s}_\lambda$ is type-full and $\mathfrak{K}^{\mathfrak{s}} = \mathfrak{K}$.

The proof is broken, as in other cases, i.e., we prove it by Theorem 2.3 which is O.K. as by 4.18 + 4.19 below its assumptions holds.

Remark. Note in the following definition FR_1, FR_2 are quite different even though $\le^1_{\mathfrak{u}}, \le^2_{\mathfrak{u}}$ are the same, except the domain.

4.17 Definition. We define $\mathfrak{u} = \mathfrak{u}^3_{\mathfrak{K}_{\aleph_0}}$ by

(a) $\partial = \partial_{\mathfrak{u}} = \aleph_1$

(b) $\mathfrak{K}_{\mathfrak{u}} = \mathfrak{K}_{\aleph_0}$ so $K^{\mathrm{up}}_{\mathfrak{u}} = \mathfrak{K}$

(c) FR_2 is the family of triples $(M, N, \mathbf{J})$ such that:

  (α) $M \le_{\mathfrak{K}} N \in \mathfrak{K}_{\aleph_0}$

  (β) $\mathbf{J} \subseteq N \backslash M$ and $|\mathbf{J}| \le 1$

(d) $(M_1, N_1, \mathbf{J}_1) \le_2 (M_2, N_2, \mathbf{J}_2)$ <u>iff</u>

  (α) both are from FR_1

  (β) $M_1 \le_{\mathfrak{K}} M_2$ and $N_1 \le_{\mathfrak{K}} N_2$

  (γ) $\mathbf{J}_1 \subseteq \mathbf{J}_2$

  (δ) if $\bar{c} \in \mathbf{J}$ then $\mathrm{gtp}(\bar{c}, M_2, N_2)$ is the stationarization of $\mathrm{gtp}(\bar{c}, M_1, N_1)$

(e) FR_1 is the class of triples $(M, N, \mathbf{J})$ such that

  (α) $M \le_{\mathfrak{K}} N$ are countable

  (β) $\mathbf{J} \subseteq N \backslash M$ or, less pedantically, $\mathbf{J} \subseteq {}^{\omega>}N \backslash {}^{\omega>}M$

  (γ) if $|\mathbf{J}| > 1$ then $\mathbf{J} = {}^{\omega>}N \backslash {}^{\omega>}M$ and N is $(\mathbf{D}(M), \aleph_0)^*$-homogeneous

(f) $\le_1$ is defined as in clause (d) <u>but</u> on FR_2.

4.18 Claim. *1) $\mathfrak{u}$ is a nice construction framework.*
2) For almost$_2$ all triples $(\bar{M}, \bar{\mathbf{J}}, \mathbf{f}) \in K^{\mathrm{qt}}_{\mathfrak{u}}$ the model M is saturated (for $\mathfrak{K}$).

Proof. 1) The main points are:

$\boxtimes_1$ $(M, N, \mathbf{J}) \in \mathrm{FR}_2$ and $n < \omega \Rightarrow (M_n, N_n, \mathbf{J}_n) \le_2 (M, N, \mathbf{J})$ <u>when</u>:

  (a) $(M_n, N_n, \mathbf{J}_n) \in \mathrm{FR}_2$

  (b) $(M_n, N_n, \mathbf{J}_n) \le_2 (M_{n+1}, N_{n+1}, \mathbf{J}_{n+1})$ for $n < \omega$

  (c) $M = \cup\{M_n : n < \omega\}$

  (d) $N = \cup\{N_n : n < \omega\}$

  (e) $\mathbf{J} = \cup\{\mathbf{J}_n : n < \omega\}$.

[<u>Why $\boxtimes_1$ holds</u>? See I.88r-5.13(9).]

$\boxtimes_2$ $(M, N, \mathbf{J}) \in \mathrm{FR}_1$ and $n < \omega \Rightarrow (M_n, N_n, \mathbf{J}_n) \le_1 (M, N, \mathbf{J})$ <u>when</u>

  (a) $(M_n, N_n, \mathbf{J}_n) \in \mathrm{FR}_1$

  (b) $(M_n, N_n, \mathbf{J}_n) \le_1 (M_{n+1}, N_{n+1}, \mathbf{J}_{n+1})$ for $n < \omega$

  (c) $M = \cup\{M_n : n < \omega\}$

  (d) $N = \cup\{N_n : n < \omega\}$

  (e) $\mathbf{J} = \cup\{\mathbf{J}_n : n < \omega\}$.

[Why does $\boxtimes_2$ holds? If $|\mathbf{J}| \le 1$ then the proof is similar to the one in $\boxtimes_1$, so assume that $|\mathbf{J}| > 1$, so as $n < \omega \Rightarrow \mathbf{J}_n \subseteq \mathbf{J}_{n+1}$ by clause (b) of $\boxtimes_2$ and $\mathbf{J} = \cup\{\mathbf{J}_n : n < \omega\}$ by clause (e) of the $\boxtimes_2$ necessarily for some $n, |\mathbf{J}_n| > 1$, so without loss of generality $|\mathbf{J}_n| > 1$ for every $n < \omega$. So by the definition of FR_1, we have:

$(*)_1$ $\mathbf{J}_n = {}^{\omega>}(N_n) \backslash {}^{\omega>}(M_n)$

$(*)_2$ N_n is $(\mathbf{D}(M_n), \aleph_0)^*$-homogeneous.

Hence easily

$(*)_3$ $\mathbf{J} = {}^{\omega>}N \backslash {}^{\omega>}M$

and as in the proof of $\boxtimes_1$ clearly

$(*)_4$ if $n < \omega$ and $\bar{c} \in \mathbf{J}_n$ then $\mathrm{gtp}(\bar{c}, M, N)$ is the stationarization of $\mathrm{gtp}(\bar{c}, M_n, N_n)$.

So the demands for "$(M_n, N_n, \mathbf{J}_n) \le_1 (M, N, \mathbf{J})$" holds except that we have to verify

$(*)_5$ N is $(\mathbf{D}(M), \aleph_0)^*$-homogeneous.

[Why this holds? Assume $N \le_{\mathfrak{K}_{\aleph_0}} N^+, \bar{a} \in {}^{\omega>}N, \bar{b} \in {}^{\omega>}(N^+)$. So $\mathrm{gtp}(\bar{a}\hat{\ }\bar{b}, M, N^+) \in \mathbf{D}(M)$, hence by I.88r-5.13(9) it is the stationarization of $\mathrm{gtp}(\bar{a}\hat{\ }\bar{b}, M_{n_0}, N^+)$ for some $n_0 < \omega$. Also for some $n_1 < \omega$, we have $\bar{a} \in {}^{\omega>}(N_{n_1})$. Now some $n < \omega$ is $\ge n_0, n_1$, so by $(*)_1 + (*)_2$ for some $\bar{b}' \in {}^{\omega>}(N_n)$ the type $\mathrm{gtp}(\bar{a}\hat{\ }\bar{b}', M_n, N_n)$ is equal to $\mathrm{gtp}(\bar{a}\hat{\ }\bar{b}, M_n, N^+)$ so $\bar{b}' \in \mathbf{J}_n$. But by $(*)_4$, also the type $\mathrm{gtp}(\bar{a}\hat{\ }\bar{b}', M, N)$ is a stationarization of $\mathrm{gtp}(\bar{a}\hat{\ }\bar{b}', M_n, N) = \mathrm{gtp}(\bar{a}\hat{\ }\bar{b}, M_n, N^+)$ hence $\mathrm{gtp}(\bar{a}\hat{\ }\bar{b}', M, N) = \mathrm{gtp}(\bar{a}\hat{\ }\bar{b}, M, N^+)$ so we are done.]
Thus we have finished proving $\boxtimes_2$]

$\boxtimes_3$ clause (F) of Definition 1.2 holds.

[Why $\boxtimes_3$? By I.88r-5.20.]
Together we have finished proving $\mathfrak{u}$ is a nice construction framework.
2) Note that

$\boxtimes_4$ $M_{\omega_1} \in K_{\aleph_1}$ is saturated <u>when</u>:

- (a) $\langle M_\alpha : \alpha \le \omega_1 \rangle$ is $\le_{\mathfrak{K}}$-increasing continuous
- (b) $\alpha < \omega_1 \Rightarrow M_\alpha \in K_{\aleph_0}$
- (c) $M_{\alpha+1}$ is $(\mathbf{D}(M_\alpha), \aleph_0)^*$-homogeneous for $\alpha < \omega_1$ or just
- $(c)'$ if $\alpha < \omega_1$ and $p \in \mathbf{D}(M_\alpha)$ is a 1-type then for some $\beta \in [\alpha, \omega_1)$ and some $c \in M_{\beta+1}$, $\mathrm{gtp}(c, M_\beta, M_{\beta+1})$ is the stationarization of p.

[Why? Obvious.]

Let $S \subseteq \omega_1$ be stationary, so clearly it suffices to prove:

$\boxtimes_5$ there is $\mathfrak{g}$ as in Definition 1.22(1), 1.23(3) for S and our $\mathfrak{u}$ such that: if $\langle(\bar{M}^\alpha, \bar{\mathbf{J}}^\alpha, \mathbf{f}^\alpha) : \alpha \le \delta\rangle$ is $\le^{\rm qs}_{\mathfrak{u}}$-increasing continuous 1-obeying $\mathfrak{g}$ (and $\delta < \partial^+$ is a limit ordinal) <u>then</u> $M^\delta \in \mathfrak{K}_{\aleph_1}$ is saturated.

[Why? Choose $\mathfrak{g}$ such that if the pair $((\bar{M}', \bar{\mathbf{J}}', \mathbf{f}'), (\bar{M}'', \bar{\mathbf{J}}'', \mathbf{f}''))$ does 1-obey $\mathfrak{g}$ then for every $\alpha < \partial$ and $p \in \mathbf{D}(M''_\alpha)$ we have

$(*)$ for stationarily many $\delta \in S$ for some $i < \mathbf{f}''(\delta)$ the type ${\rm gtp}(a, M''_\alpha, M''_{\delta+i+1})$ is the stationarization of p where $\mathbf{J}''_{\delta+i} = \{a\}$.

Now assume that $\langle(\bar{M}^\zeta, \bar{\mathbf{J}}^\zeta, \mathbf{f}^\zeta) : \zeta < \delta\rangle$ is $\le_{\rm qs}$-increasing (for our present $\mathfrak{u}$) and $\delta = \sup(u)$ where $u = \{\zeta : ((\bar{M}^\zeta, \mathbf{J}^\zeta, \mathbf{f}^\zeta), (\bar{M}^{\zeta+1}, \bar{\mathbf{J}}^{\zeta+1}, \mathbf{f}^{\zeta+1}))$ does 1-obeys $\mathfrak{g}\}$, and we should prove that $M^\delta := \cup\{M^\zeta_\partial : \zeta < \partial\}$ is saturated. Without loss of generality u contains all odd ordinals $< \delta$ and $\delta = {\rm cf}(\delta)$. If $\delta = \aleph_1$ this is obvious, and if $\delta = \aleph_0$ just use non-forking of types, and the criterion in $\boxtimes_4$ using $(*)$. So $\boxtimes_5$ is proved.] $\square_{4.18}$

4.19 Claim. $\mathfrak{u}$ *has the weak coding property.*

Proof. Clearly by clause $(d)'$ of the assumption, i.e. by Definition I. 88r-5.21 (2),(3) there are $N_\ell\,(\ell \le 2)), N'_3, N''_3$ such that:

$(*)_1$ (a) $N_0 \le_{\mathfrak{u}} N_1$ and $N_0 \le_{\mathfrak{u}} N_2$

(b) $N_\ell \le_{\mathfrak{u}} N'_3$ and $N_\ell \le_{\mathfrak{u}} N''_3$ for $\ell = 1, 2$

(c) N_0, N_1, N_2, N'_3 is in one-sided amalgamation, i.e. $\bar{a} \in {}^{\omega>}(N_1) \Rightarrow (N_0, N_1, \{\bar{a}\}) \le^2_{\mathfrak{u}} (N_2, N'_3, \{\bar{a}\})$ (hence $N_1 \cap N_2 = N_0$)

(d) N_0, N_1, N_2, N''_3 is in one sided amalgamation

(e) there are no (N_3, f) such that $N''_3 \le_{\mathfrak{u}} N_3$ and f is a $\le_{\mathfrak{u}}$-embedding of N'_3 into N_3 over $N_1 \cup N_2$.

Now without loss of generality

$(*)_2$ $(N_0, N_1, \mathbf{J}) \in {\rm FR}_1$ where $\mathbf{J} \in {}^{\omega>}(N_1) \backslash {}^{\omega>}(N_0)$ such that $|\mathbf{J}| > 1$.

[Why? We can find $N_1^+ \in K_{\mathfrak{u}}$ which is $(\mathbf{D}(N_1), \aleph_0)^*$-homogeneous over N_1 and without loss of generality $N_1^+ \cap N_3' = N_1 = N_1^+ \cap N_3''$. Now we can find N_3^* and $N_3^{**} \in K_{\mathfrak{u}}$ such that $(N_1, N_1^+, N_3', N_3^*)$ as well as $(N_1, N_1^+, N_3'', N_3^{**})$ is in one sided stable amalgamation. It follows that $(N_0, N_1^+, N_2, N_3^*, N_3^{**})$ satisfies all the requirements on $(N_0, N_1, N_2, N_3', N_3'')$ and in addition the demand in $(*)_2$ so we are done.]

Also without loss of generality

$(*)_3$ $(N_0, N_2, \mathbf{I}) \in$ FR_2 and $|\mathbf{I}| = 1$ and N_2 is $(\mathbf{D}(N_0), \aleph_0)^*$-homogeneous.

[Why? Similarly to $(*)_1$.]

To prove $\mathfrak{u}$ has the weak coding we can assume (the saturation is justified by 4.18(2))

$(*)_4$ $(\bar{M}, \bar{\mathbf{J}}, \mathbf{f}) \in K_{\mathfrak{u}}^{\mathrm{qt}}$ and $M := \cup\{M_\alpha : \alpha < \omega_1\}$ is saturated for $\mathfrak{K}$.

Now by renaming without loss of generality

$(*)_5$ $N_0 \leq_{\mathfrak{K}} M_{\alpha(0)}$ and $N_1 \cap M = N_0$ and $(N_0, N_1, \mathbf{J}) \leq_1 (M_{\alpha(0)}, N_1', \mathbf{J}')$ and $N_1' \cap M = M_{\alpha(0)}$.

It suffices to prove that $(\alpha(0), N_1', \mathbf{J}')$ is as required in 2.2(3). Next by the definition of "having the weak coding property", for our purpose we can assume we are given $(N'', \mathbf{J}'')$ such that

$(*)_6$ $\alpha(0) \leq \delta < \omega_1$ and $(N_0, N_1', \mathbf{J}') \leq_1 (M_\delta, N'', \mathbf{J}'')$.

By the definition of $\leq_1$ we know that

$(*)_7$ N'' is $(\mathbf{D}(M_\delta), \aleph_0)^*$-homogeneous over M_δ.

As $\cup\{M_\alpha : \alpha < \omega_1\}$ is saturated (for $K^{\mathfrak{u}}$) we can find $\beta \in (\delta, \omega_1)$ such that M_β is $(\mathbf{D}(M_\delta), \aleph_0)^*$-homogeneous over M_δ.

As $\mathfrak{K}$ is categorical in $\aleph_0$

$(*)_8$ there is an isomorphism f_0 from N_0 onto M_δ.

Similarly using the uniqueness over N_0 of a countable $(\mathbf{D}(M_0), \aleph_0)^*$-homogeneous model over N_0

$(*)_9$ there are isomorphisms f_1, f_2 from N_1, N_2 onto N'', M_β respectively extending f_0.

Lastly, M_β, N can be amalgamated over M_δ in the following two ways:

- $\odot_1$ there are $f', M' \in K_{\mathfrak{u}}$ such that f' is an isomorphism from N'_3 onto M' extending $f_1 \cup f_2$
- $\odot_2$ there are $f'', M'' \in K_{\mathfrak{u}}$ such that f'' is an isomorphism from N^1_3 onto M'' extedning $f_1 \cup f_2$.

This is clearly enough. The rest should be clear.] $\square_{4.19}$

Proof of 4.16. By the claims above. $\square_{4.16}$

* * *

<u>(D) Density of $K^{3,\mathrm{uq}}_\lambda$ when minimal triples are dense</u>

Having taken care of VI§3,§4 and of I§5, we now deal with proving the non-structure results of VI§6, i.e. [Sh 576, §6], relying on 2.3 instead of [Sh 576, §3]. Of course, later we prove stronger results but have to work harder, both model theoretically (including "$\mathfrak{s}$ is almost a good λ-frame") and set theoretically (using (vertical coding in) Theorem 2.11 and §3 rather than (weak coding in) Theorem 2.3).

This is used in VI. E46-6f.13.

4.20 Theorem. *The non-structure results of VI. E46-6f.13, Case 1 holds.*

It details: $\dot{I}(\lambda^{++}, \mathfrak{K}) \geq \mu_{\mathrm{unif}}(\lambda^{++}, 2^{\lambda^+})$ <u>when</u> we are assuming:

- (A) *set theoretically:*
  - (a) $2^\lambda < 2^{\lambda^+} < 2^{\lambda^{++}}$ *and*
  - (b) *the weak diamond ideal on $\partial := \lambda^+$ is not ∂^+-saturated*
- (B) *model theoretically:*
  - (a) *$\mathfrak{K}$ is an a.e.c., $\lambda \geq \mathrm{LS}(\mathfrak{K})$*
  - (b) (α) *$\mathfrak{K}$ is categorical in λ*
    - (β) *$\mathfrak{K}$ is categorical in λ^+ or just has a superlimit model in λ^+*
  - (c) (α) *$\mathfrak{K}$ has amalgamation in λ*
    - (β) *$\mathfrak{K}$ is stable in λ <u>or</u> just $M \in K_\lambda \Rightarrow |\mathscr{S}^{\min}_{\mathfrak{K}}(M)| \leq \lambda$*
  - (d) (α) *the minimal types are dense (for $M \in \mathfrak{K}_\lambda$)*

(β) *for $M \in K_\lambda$ the set $\mathscr{S}^{\min}_{\mathfrak{K}_\lambda}(M) = \{p \in \mathscr{S}_{\mathfrak{K}_\lambda}(M) : p$ minimal$\}$ is inevitable*

(γ) *the $M \in K^{\rm slm}_{\lambda^+}$ is saturated above λ*

(e) *above (by $\leq_{\rm na}$) some $(M^*, N^*, a) \in K^{3,{\rm na}}_\lambda$ there is no triple with the uniqueness property, i.e. from $K^{3,{\rm uq}}_\lambda$, see VI.E46-6f.2.*

4.21 Remark. 1) Note: every $M \in K_{\lambda^+}$ is saturated above λ when the first, stronger version of (B)(b)(β) holds noting (B)(c)(β)+ (B)(d)(β).
2) When we use the weaker version of clause (b)(β), i.e. "there is superlimit $M \in K_{\lambda^+}$" then we have to prove that for almost$_2$ every $(\bar M, \bar{\mathbf{J}}, \mathbf{f})$, the model M_{λ^+} is saturated above λ which, as in earlier cases, can be done; see VI.E46-2b.13(4).
3) Concerning clause (B)(d): "the minimal types are dense", it follows from $(\rm amg)_\lambda+ (\rm stb)_\lambda$, i.e. from clause (c) recalling VI.E46-2b.4(4).
4) Note that 4.23, 4.24 does not depend on clause (A)(b) of 4.20.

4.22 Definition. We define $\mathfrak{u} = \mathfrak{u}_4 = \mathfrak{u}^4_{\mathfrak{K}_\lambda}$ as follows:

(a) $\partial_{\mathfrak{u}} = \lambda^+$

(b) $\mathfrak{K}_{\mathfrak{u}} = \mathfrak{K}_\lambda$ or pedantically $\mathfrak{K}'_\lambda$, see Definition 1.10

$(c)_1$ ${\rm FR}^{\mathfrak{u}}_1$ is the set of triples $(M, N, \mathbf{I})$ satisfying $M \leq_{\mathfrak{K}} N \in K_\lambda, \mathbf{I} = \emptyset$ or $\mathbf{I} = \{a\}$ and the type $\mathbf{tp}_{\mathfrak{K}_\lambda}(a, M, N)$ is minimal, pedantically, if $a/=^N \notin M/=^N$ then $\mathbf{tp}(a/=^N, M/=^N, N/=^N)$ is minimal

$(c)_2$ $(M_1, N_1, \mathbf{I}_1) \leq_1 (M_2, N_2, \mathbf{I}_2)$ iff (both are ${\rm FR}^{\mathfrak{u}}_1$ and) $M_1 \leq_{\mathfrak{K}} M_2, N_1 \leq_{\mathfrak{K}} N_2$ and $\mathbf{I}_1 \subseteq \mathbf{I}_2$ (in the non-trivial cases, equivalently, $\mathbf{I}_1 = \mathbf{I}_2$), pedantically, if $(a/=^N) \notin M/=^N$ then $\mathbf{tp}(a/=^N, M/=^N, N/=^N)$ is minimal

(d) ${\rm FR}^{\mathfrak{u}}_2 = {\rm FR}^{\mathfrak{u}}_1$ and $\leq^2_{\mathfrak{u}} = \leq^1_{\mathfrak{u}}$.

4.23 Claim. $\mathfrak{u}$ *is a nice construction framework which is self-dual.*

Proof. Easy (amalgamation, i.e. clause (F) of Definition 1.2 holds by the proof of symmetry in Axiom (E)(f) in proof of Theorem VI.E46-8h.1). $\square_{4.23}$

4.24 Claim. *1) Every $(M, N, \mathbf{I}) \in \mathrm{FR}_1$ such that $[a \in \mathbf{I} \ \ \& \ \ b \in M \Rightarrow \neg a =^N_\tau b]$ has the true weak coding property (see Definition 2.2(1A)).*
2) $\mathfrak{u}$ has the weak coding property.
3) For almost$_2$ every $(\bar M, \bar{\mathbf{J}}, \mathbf{f}) \in K^{\rm qt}_{\mathfrak{u}}$ the model $M_\partial \in K_{\lambda^+}$ is satuarted above λ.
4) For some $\mathfrak{u} - \{0,2\}$-appropriate function $\mathfrak{h}$, for every $M \in K^{\mathfrak{u},\mathfrak{h}}_{\lambda^{++}}$ the model $M/=^M$ has cardinality λ^{++} and is saturated above λ.

Proof. 1) Straight.
2) By part (1) above and part (3) below.
3) By clauses (B)(c)$(\alpha),(\beta)$ of 4.20, clearly there is a $M \in K_{\lambda^+}$ which is saturated above λ. If in (B)(b)(β) we assume categoricity in λ^+ then every $M \in K_{\lambda^+}$ is saturated above λ, but then it is obvious that part (1) implies part (2) by 2.10(4)(b). For any stationary $S \subseteq \partial$, we choose $\mathfrak{h}$ such that

- $(*)$ if $((\bar M^1, \bar{\mathbf{J}}^1, \mathbf{f}^1), (\bar M^2, \bar{\mathbf{J}}^2, \mathbf{f}^2))$ does 2-obeys $\mathfrak{h}$ then: for stationarily many $\delta \in S$ there is successor ordinal $i < \mathbf{f}^2(\delta)$ such that $M^2_{\delta+i+1}$ is $<_{\mathfrak{K}_\lambda}$-universal over $M^2_{\delta+i}$ (hence M^2_∂ is saturated above λ and is the superlimit model in $\mathfrak{K}_{\lambda^+}$).

Alternatively do as in 4.14(4), using VI.E46-8h.1.
4) As in 4.7(3).

Proof of 4.20. By 4.23, 4.24 and Theorem 2.3. $\square_{4.20}$

* * *

(E) Density of $K^{3,\mathrm{uq}}_{\mathfrak{s}}$ for good λ-frames

We now deal with the non-structure proof in III.600-nu.6, that is justifying why the density of $K^{3,\mathrm{uq}}_{\mathfrak{s}}$ holds.

Before we state the theorem, in order to get rid of the problem of disjoint amalgamation, one of the ways is to note:

4.25 Definition. Assume that $\mathfrak{s}$ is a good λ-frame (or just an almost good λ-frame see Definition in 5.2 below or just a pre-λ-frame, see VI.E46-8h.3).
1) We say that $\mathfrak{s}$ has fake equality $=_*$ when $\mathfrak{K}_{\mathfrak{s}}$ has the fake equality $=_*$, see Definition 3.17(1) and $\mathbf{tp}_{\mathfrak{s}}(a, M_1, M_2)$ does not fork over M_0 iff $M_0 \le_{\mathfrak{s}} M_1 \le_{\mathfrak{s}} M_2, a \in M_2$ and letting $M'_\ell = M_\ell/=^{M_2}_*$ we have $(\forall b \in M_1)(\neg(a =^{M_2}_* b)) \Rightarrow$

$\mathbf{tp}_{\mathfrak{s}}(a/=_*^{M_2}, M_1', M_2')$ does not fork over M_0'.
2) We define $\mathfrak{s}' = (K_{\mathfrak{s}'}, \mathscr{S}^{\rm bs}_{\mathfrak{s}'}, \bigcup\limits_{\mathfrak{s}'})$ as follows:

- (a) $\mathfrak{K}_{\mathfrak{s}'} = K'_{\mathfrak{s}}$, see 1.10 as in 4.5 so $\tau_{\mathfrak{s}'} = \tau'_{\mathfrak{s}} = \tau_{\mathfrak{s}} \cup \{=_\tau\}$ and a $\tau'_{\mathfrak{s}}$-model M belongs to $K_{\mathfrak{s}'}$ iff $=_*^M$ is a congruence relation and the model $M/=_\tau^{M'}$ belongs to $\mathfrak{K}_{\mathfrak{s}}$
- (b) for $M' \in K_{\mathfrak{s}'}$ we let $\mathscr{S}^{\rm bs}_{\mathfrak{s}'}(M') = \{\mathbf{tp}_{\mathfrak{K}_{\mathfrak{s}'}}(a, M', N') : M' \le_{\mathfrak{s}'} N'$ and $\mathbf{tp}_{\mathfrak{s}}(a/=_\tau^{N'}, M'/=_\tau^{M'}, N'/=_\tau^{N'}) \in \mathscr{S}^{\rm bs}_{\mathfrak{s}}(M'/=_\tau^{M'})$ or $a \in N' \backslash M'$ but $(\exists b \in M')(a =_\tau b)\}$
- (c) $\mathbf{tp}_{\mathfrak{s}'}(a, M_1', M_2')$ does not fork over M_0' when $M_0' \le_{\mathfrak{s}'} M_1' \le_{\mathfrak{s}'} M_2'$ and either $\mathbf{tp}_{\mathfrak{s}}(a/=_\tau^{M_2'}, M_1' =_\tau^{M_2'}, M_2'/=_\tau^{M_2'})$ does not fork over $M_0'/=^{M_2'}$ or for some $b \in M_0'$ we have $M_2' \models$ "$a =_\tau b$" but $a \notin M_1'$.

4.26 Claim. *Let $\mathfrak{s}, \mathfrak{s}'$ be as in 4.25(2).*
1) If $\mathfrak{s}$ is a good λ-frame <u>then</u> $\mathfrak{s}'$ is a good λ-frame and if $\mathfrak{s}$ is an almost good λ-frame then $\mathfrak{s}'$ is an almost good λ-frame; and if $\mathfrak{s}$ is a pre-λ-frame then $\mathfrak{s}'$ is a pre-λ-frame. In all cases $\mathfrak{s}'$ has the fake equality $=_\tau$.
2) For $\mu \ge \lambda, \dot I(\mu, K^{\mathfrak{s}}) = |\{M'/\cong : M' \in K^{\mathfrak{s}'}_\mu$ and is $=_\tau$-fuller, that is $a \in M' \Rightarrow \|M'/=_\tau^{M'}\| = \mu = |\{b \in M' : a =_\tau^{M'} b\}|\}$.
3) If $M' \in K^{\mathfrak{s}'}$ <u>then</u> M' is λ^+-saturated above λ for $\mathfrak{s}'$ iff $M'/=_\tau^{M'}$ is λ^+-saturated above λ for $\mathfrak{s}$ and M' is $(\lambda^+, =_\tau)$-full (recalling 1.10(5A)).

4.27 Remark. 1) By 4.26(2), the proof of "$\dot I(\mu, K^{\mathfrak{s}'})$ is $\ge \chi$" here usually gives "$\dot I(\mu, K^{\mathfrak{s}})$ is $\ge \chi$".
2) We define $\mathfrak{s}'$ such that for some 0-appropriate $\mathfrak{h}$, if $\langle(\bar M^\alpha, \bar{\mathbf{J}}^\alpha, \mathbf{f}^\alpha) : \alpha < \partial^+\rangle$ is $\le_{\rm qt}$-increasing continuous 0-obeying $\mathfrak{h}$, then $M = \cup\{M^\alpha_\partial : \alpha < \partial^+\}$ satisfies the condition in 4.26(2); it does not really matter if we need $\{0,2\}$-appropriate $\mathfrak{h}$.
3) Recall Example 1.12 as an alternative to 4.26(2).
4) Another way to deal with disjointness is by 5.22, 5.23 below.

Proof. Easy and see 1.11. $\square_{4.26}$

4.28 Theorem. *Like 4.20 but dealing with $\mathfrak{s}$, i.e. replacing clause (B) by clause (B)′ stated below; that is, $\dot I(\lambda^{++}, K^{\mathfrak{s}}) \ge \mu_{\rm unif}(\lambda^{++}, 2^\lambda)$ <u>when</u>:*

- (A) *set theoretically:*
  - (a) $2^\lambda < 2^{\lambda^+} < 2^{\lambda^{++}}$ *and*
  - (b) *the weak diamond ideal on $\partial := \lambda^+$ is not ∂^+-saturated*

$(B)'$ *model theoretic*

- (a) *$\mathfrak{s}$ is a good λ-frame (or just an almost good λ-frame, see 5.2) with $\mathfrak{K}_{\mathfrak{s}} = \mathfrak{K}_\lambda$*
- (b) *density of $K^{3,\mathrm{uq}}_{\mathfrak{s}}$ fail, i.e. for some $(M,N,a) \in K^{3,\mathrm{bs}}_{\mathfrak{s}}$ we have $(M,N,a) \le_{\mathrm{bs}} (M',N',a) \Rightarrow (M',N',a) \notin K^{3,\mathrm{uq}}_{\mathfrak{s}}$, see Definition 5.3.*

Proof. We apply 2.3, its assumption holds by Definition 4.29 and Claim 4.30 below applied to $\mathfrak{s}'$ from 4.25 by using 4.26.

4.29 Definition. Let $\mathfrak{s}$ be as in 4.28 (or just a pre-λ-frame). We let $\mathfrak{u} = \mathfrak{u}_{\mathfrak{s}} = \mathfrak{u}^1_{\mathfrak{s}}$ be defined as

- (a) $\partial_{\mathfrak{u}} = \lambda^+_{\mathfrak{s}}$
- (b) $\mathfrak{K}_{\mathfrak{u}} = \mathfrak{K}_{\mathfrak{s}}$
- (c) $\mathrm{FR}^{\mathfrak{u}}_1 = \{(M,N,\mathbf{I}) : M \le_{\mathfrak{K}_{\mathfrak{u}}} N, \mathbf{I} = \emptyset$ or $\mathbf{I} = \{a\}$ where $a \in N$ and $\mathbf{tp}_{\mathfrak{s}}(a,M,N) \in \mathscr{S}^{\mathrm{bs}}_{\mathfrak{s}}(M)\}$
- (d) $\le^1_{\mathfrak{u}}$ is defined by $(M_1,N_1,\mathbf{I}_1) \le^1_{\mathfrak{u}} (M_2,N_2,\mathbf{I}_2)$ <u>when</u> both are from $\mathrm{FR}^{\mathfrak{u}}_1$, $M_1 \le_{\mathfrak{s}} M_2, N_1 \le_{\mathfrak{s}} N_2, \mathbf{I}_1 \subseteq \mathbf{I}_2, M_1 = M_2 \cap N_1$ and if $\mathbf{I}_1 = \{a\}$ then $\mathbf{tp}_{\mathfrak{s}}(a,M_2,N_2)$ does not fork (for $\mathfrak{s}$) over M_1 (so if $\mathbf{I}_2 = \{a_\ell\}$ for $\ell = 1,2$ this means $(M_1,N_1,a_1) \le_{\mathrm{bs}} (M_2,N_2,a_1)$)
- (e) $\mathrm{FR}^{\mathfrak{u}}_2 = \mathrm{FR}^{\mathfrak{u}}_1$ and $\le^2_{\mathfrak{u}} = \le^1_{\mathfrak{u}}$.

4.30 Claim. *Let $\mathfrak{u} = \mathfrak{u}_{\mathfrak{s}'}$ where $\mathfrak{s}'$ is from Definition 4.25 or $\mathfrak{u} = \mathfrak{s}$ except when we mention equality (or $=_\tau$-fuller).*
1) $\mathfrak{u}$ is a nice construction framework which is self dual.
2) For almost$_2$ every $(\bar M, \bar{\mathbf{J}}, \mathbf{f}) \in K^{\mathrm{qt}}_{\mathfrak{u}}$ the model $M := \cup\{M_\alpha : \alpha < \lambda^+\}$ is saturated, see Definition 1.22(3C), see 4.3.
3) $\mathfrak{u}$ has the weak coding property.
4) There is a $\mathfrak{u}$-0-appropriate function $\mathfrak{h}$ such that every $M \in K^{\mathfrak{u},\mathfrak{h}}_{\lambda^{++}}$ is λ^+-saturated above λ and is $=_\tau$-fuller (hence $M/=^M_\tau$ has cardinality λ^{++}).
5) Moreover, there is a $\mathfrak{u}-\{0,2\}$-appropriate function $\mathfrak{h}$ such that if $\langle(\bar M^\alpha, \bar{\mathbf{J}}^\alpha, \mathbf{f}^\alpha) : \alpha < \lambda^{++}\rangle$ obeys $\mathfrak{h}$ <u>then</u> for some club E of λ^{++} the model M^δ_∂ is saturated above λ for $\delta \in E$ and $\cup\{M^\zeta_\partial : \zeta < \lambda^{++}\}$ is $=_\tau$-fuller.
6) Also $\mathfrak{u}$

(α) *satisfies* $(E)_\ell(e)$*, monotonicity (see 1.13(1))*

(β) *is hereditary (see Definition 3.17(2),(3))*

(γ) *if* $\mathfrak{u} = \mathfrak{u}_{\mathfrak{s}}, \mathfrak{s}$ *from 4.25(2) then* $=_\tau$ *is a fake equality for* $\mathfrak{u}$*, (see Definition 3.16(1))*

(δ) $\mathfrak{u}$ *is hereditary for the fake equality* $=_\tau$*, (see Definition 3.17(4))*

(ε) $\mathfrak{u}$ *is interpolative, see Definition 3.21.*

4.31 Remark. 1) In claim 5.11 we shall deal with the almost good case, (see Definition 5.2), the proof below serves there too.
2) In 4.30, only clause $(B)'(a)$ from the assumptions of Theorem 4.28 is used except in part (3) which uses also clause (B)′(b).
3) Part (6) of 4.30 is used only in 6.19.
4) Most parts of 4.30 holds also for $\mathfrak{u} = \mathfrak{u}_{\mathfrak{s}}$, i.e. we have to omit the statements on $=_\tau$-fuller, fake equality.

Proof. 1) Note

<u>Clause $(D)_\ell(d)$</u>: Given $M \in K_{\mathfrak{s}}$, it is not $<_{\mathfrak{s}}$-maximal hence there is N such that $M <_{\mathfrak{s}} N$ hence by density (Ax(D)(c) of (almost) good λ-frames) there is $c \in N$ such that $\mathbf{tp}_{\mathfrak{s}}(c, M, N) \in \mathscr{S}^{\rm bs}_{\mathfrak{s}}(M)$, so $(M, N, a) \in \mathrm{FR}^+_1$, as required.

<u>Clause $(E)_\ell(c)$</u>: Preservation under increasing union.

Holds by axiom $(E)(h)$ of Definition of III.600-1.1 of $\mathfrak{s}$ being a good λ-frame (and similarly for being an almost good λ-frame).

<u>Clause (F), amalgamation</u>:

This holds by symmetry axiom $(E)(i)$ of Definition III.600-1.1 of $\mathfrak{s}$ being a good λ-frame (and similarly for $\mathfrak{s}$ being an almost good λ-frame). The disjointness is not problematic in proving clause (F) of Definition 1.2 because

$(*)_1$ for $\mathfrak{u} = \mathfrak{u}_{\mathfrak{s}}$ we can prove it (when $\mathfrak{K}$ is categorical in λ, see 5.23, 5.12 below) and it follows by our allowing the use of $=_\tau$ or use $\mathfrak{s}'$ (see 4.30).

2) We just use

$(*)$ M is saturated ($\in \mathfrak{K}_{\lambda^+}$) when

(a) $M = \cup\{M_\alpha : \alpha < \lambda^+\}$

(b) $M_\alpha \in K_{\mathfrak{s}}$ is $\leq_{\mathfrak{s}}$-increasing continuous

(c) if $p \in \mathscr{S}^{\rm bs}_{\mathfrak{s}}(M_\alpha)$ then for some $\beta \in [\alpha, \lambda^+)$ the non-forking extension $q \in \mathscr{S}^{\rm bs}_{\mathfrak{s}}(M_\beta)$ of p is realized in $M_{\beta+1}$ (or just in some $M_\gamma, \gamma \in (\beta, \lambda^+)$).

See III§4; more fully see the proof of part (5).
3) Let $(M,N,a) \in K^{3,\mathrm{bs}}_{\mathfrak{s}}$ be such that there is no triple $(M',N',a) \in K^{3,\mathrm{uq}}_{\mathfrak{s}}$ which is $<_{\mathrm{bs}}$-above it, exists by clause (B)$'$(b) from Theorem 4.28. Let $\mathbf{I} = \{a\}$, so if $(M,N,\mathbf{I}) \le^1_{\mathfrak{u}} (M',N',\mathbf{I}')$ then ($\mathbf{I}' = \mathbf{I} = \{a\}$ and) $(M',N',a) \in K^{3,\mathrm{bs}}_{\mathfrak{s}} \backslash K^{3,\mathrm{uq}}_{\mathfrak{s}}$ hence there are M'',N_1,N_2 such that $(M',N',\mathbf{I}) \le^1_{\mathfrak{u}} (M'',N_\ell,\mathbf{I})$ and N_1,N_2 are $\le_{\mathfrak{s}}$-incompatible amalgamations of M'',N' over M'. This shows that $(M',N',\mathbf{I})$ has the true weak coding property. As for almost$_2$ every triple $(\bar M,\bar{\mathbf{J}},\mathbf{f}) \in K^{\mathrm{qt}}_{\mathfrak{s}}, M_\partial = M_{\lambda^+}$ is saturated, by 2.10(4) and part (5) we get that $\mathfrak{u}$ has the weak coding property.
4) Easy to check by 1.11 or as in (5).
5) We choose $\mathfrak{h}$ such that:

- $\boxtimes$ if $\mathbf{x} = \langle (\bar M^\zeta,\bar{\mathbf{J}}^\zeta,\mathbf{f}^\zeta) : \zeta \le \zeta(*)\rangle$ is $\le_{\mathrm{qt}}$-increasing continuous and obey $\mathfrak{h}$ is $\xi < \zeta(*)$ <u>then</u>
  - (α) $(\bar M^\xi,\bar{\mathbf{J}}^\xi,\mathbf{f}^\xi) <^{\mathrm{at}}_{\mathfrak{u}} (\bar M^{\xi+1},\bar{\mathbf{J}}^{\xi+1},\mathbf{f}^{\alpha+1})$ and let it be witnessed by $E,\bar{\mathbf{I}}$
  - (β) $M^{\xi+1}_{\delta+1}$ is brimmed over M^ξ_δ for a club of $\delta < \lambda^+$
  - (γ) if $\zeta \le \xi$ is minimal such that one of the cases occurs, then the demand in the first of the cases below holds:

<u>Case A</u>: There is $a \in M^\xi_\partial$ such that $a/=^{M^\xi}_\tau$ is $\subseteq M^\zeta_\partial$ and $\zeta < \xi$.

<u>Then</u> for some such $a', M^{\xi+1}_\partial \models$ "$a' =_\tau b$" (but $b \notin M^\zeta_\partial$), in fact $b' \in \mathbf{I}_\alpha$ for some $\alpha < \partial$ large enough.

<u>Case B</u>: $\zeta < \xi$, not Case A (for ζ) but for some $\alpha < \partial$ and $p \in \mathscr{S}^{\mathrm{bs}}_{\mathfrak{s}}(M^\zeta_\alpha)$ for no $\varepsilon \in [\zeta,\xi)$ are there $a \in M^{\varepsilon+1}_\partial$ such that $\mathbf{tp}_{\mathfrak{s}}(a,M^\varepsilon_\beta,M^{\varepsilon+1}_\beta)$ is a non-forking extension of p for every $\beta < \partial$ large enough.

<u>Then</u> for some such (p,α) we have $\mathbf{tp}_{\mathfrak{s}}(b,M^\zeta_\beta,M^{\zeta+1}_\partial)$ is a non-forking extension of p for every $\beta < \partial$ large enough.

<u>Case C</u>: $\zeta < \xi$, Cases A,B fail for ζ and there is a pair (α,p) such that $\alpha < \partial, p \in \mathscr{S}^{\mathrm{bs}}_{\mathfrak{s}}(M^\zeta_\alpha)$ such that for no $\varepsilon \in [\zeta,\xi)$ is the set $S := \{\delta < \partial$: there is $i < \mathbf{f}^{\varepsilon+1}(\delta)$ such that $\mathbf{tp}_{\mathfrak{s}}(a_{\mathbf{J}^{\varepsilon+1}_{\delta+i}},M^{\varepsilon+1}_{\delta+i},M^{\varepsilon+1}_{\delta+i+1})$ is a non-forking extension of $p\}$ stationary.

<u>Then</u> for some such pair (α,p), the condition above holds for ξ.

<u>Case D</u>: $\zeta = \xi$.

Does not matter.
6) Easy, too. $\square_{4.30}$

* * *

<u>(F) The better versions of the results</u>:

Here we prove the better versions of the results, i.e. without using on "WDmId(λ^+) is λ^{++}-saturated" but relying on later sections.

Of course, the major point is reproving the results of §4(E), i.e. "non-structure for a good λ-frame $\mathfrak{s}$ failing the density of $K^{3,\mathrm{uq}}_{\mathfrak{s}}$", we have to rely on §5-§8.

We also deal with §4(D); here we rely on VI§8, so we get an almost good λ-frame $\mathfrak{s}$ (rather than good λ-frames). But in §5-§8 we deal also with this more general case (and in §7, when we discard a non-structure case, we prove that $\mathfrak{s}$ is really a good λ-frame).

Lastly, we revisit §4(C).

4.32 Theorem. *1) In Theorem 4.28 we can omit the assumption (A)(b).*
2) $\dot I(\lambda^{++},\mathfrak{K}) \ge \mu_{\mathrm{unif}}(\lambda^{++},2^{\lambda^+})$ and moreover $\dot I(\lambda^{++},K^{\mathfrak{u},\mathfrak{h}}) \ge \mu_{\mathrm{unif}}(\lambda^{++},2^{\lambda^+})$ for $\mathfrak{u}=\mathfrak{u}^1_{\mathfrak{s}}$ from Definition 4.29 or $\mathfrak{u}=\mathfrak{u}^3_{\mathfrak{s}}$ from Definition 8.3 and any $\mathfrak{u}-\{0,2\}$-appropriate function $\mathfrak{h}$, <u>when</u>:

- (A) *(set theoretic assumption), $2^\lambda < 2^{\lambda^+} < 2^{\lambda^{++}}$*
- (B) *(model theoretic assumptions),*
  - (a) *$\mathfrak{s}$ is an almost good λ-frame*
  - (b) *$\mathfrak{s}$ is categorical in λ*
  - (c) *$\mathfrak{s}$ is not a good λ-frame <u>or</u> $\mathfrak{s}$ (is a good λ-frame which) fail density for $K^{3,\mathrm{uq}}_{\mathfrak{s}}$.*

4.33 Remark. 1) This proves VI. E46-0z.1 from VI. E46-8h.9, proving the main theorem VI. E46-0z.1.
2) We can phrase the theorem also as: if (A),(B)(a),(B)(b) holds and the $\dot I(K^{\mathfrak{u},\mathfrak{h}}_{\lambda^{++}}) < \mu_{\mathrm{unif}}(\lambda^{++},2^\lambda)$ <u>then</u> $\mathfrak{s}$ is a good λ-frame for which $K^{3,\mathrm{uq}}_{\mathfrak{s}}$ is dense in $(K^{3,\mathrm{bs}}_{\mathfrak{s}},\le_{\mathrm{bs}})$ (and so has existence for $K^{3,\mathrm{uq}}_{\mathfrak{s}}$).

Proof. 1) This is a special case of part (2).
2) Toward contradiction assume that the desired conclusion fail.

First, the Hypothesis 5.1 of §5 holds for $\mathfrak{s}$ hence its results. Second, the Hypothesis 6.1 of §6 apply hence its results. So consider conclusion 6.17(2); its assumption "$2^{\lambda^+} < 2^{\lambda^{++}}$" holds by assumption (A) here, and its assumption

"$\dot I(K^{\mathfrak{s},\mathfrak{h}}_{\lambda^{++}}) < \mu_{\rm unif}(\lambda^{++},2^{\lambda^+})$ for $\mathfrak{u} = \mathfrak{u}^1_{\mathfrak{s}}$ for some $\mathfrak{u} - \{0,2\}$-appropriate $\mathfrak{h}$" holds by our present assumption toward contradiction and its assumption "$\mathfrak{K}_{\mathfrak{s}}$ is categorical" holds by clause (B)(b) of the assumption of 4.32.

Hence the conclusion of 6.17 holds which says that

$(*)$ $\mathfrak{s}$ has existence for $K^{3,\rm up}_{\mathfrak{s},\xi}$ for every $\xi \le \lambda^+$, see Definition 6.4.

Now consider Hypothesis 7.1; now part (1) there ($\mathfrak{s}$ is an almost good λ-frame) holds by the present assumption (B)(a), part (2) there was just proven; part (3) there ($\mathfrak{s}$ is categorical in λ) holds by the present assumption (B)(b), and lastly, part (4) there (disjointness) is proved in 5.23. So Hypothesis 7.1 of §7 holds hence the results of that section up to 7.20 apply.

In particular, $\mathrm{WNF}_{\mathfrak{s}}$ defined in 7.3(1),(2) is well defined and by 7.17(1) is a weak non-forking relation on ${}^4(K_{\mathfrak{s}})$ respecting $\mathfrak{s}$. Also $\mathfrak{s}$ is a good λ-frame by Lemma 7.19(1) so the first possibility in clause (B)(c) of 4.32 does not hold. By inspection all parts of Hypothesis 8.1 of §8 holds hence the results of that section apply.

Now in Claim 8.19, its conclusion fails as this means our assumption toward contradiction and among its assumptions, clause (a), saying "$2^\lambda < 2^{\lambda^+} < 2^{\lambda^{++}}$" holds by clause (A) of 4.32, clause (c) saying "K is categorical in λ" holds by clause (B)(b) of 4.32 and clause (d) saying "$\mathfrak{u} = \mathfrak{u}^3_{\mathfrak{s}}$ from 8.3 has existence for $K^{3,\rm up}_{\mathfrak{s},\lambda^+}$" was proved above. So clause (b) of 8.19 fails, i.e. $\mathfrak{s}$ fails the non-uniqueness for $\mathrm{WNF}_{\mathfrak{s}}$, but by 8.12(1) this implies that we have uniqueness for WNF.

Lastly, we apply Observation 8.12(2), it has two assumptions, the first "$\mathfrak{s}$ has existence for $K^{3,\rm up}_{\mathfrak{s},\lambda^+}$", was proved above, and second "$\mathfrak{s}$ has uniqueness for WNF" has just been proved; so the conclusion of 8.12 holds. This means "$\mathfrak{s}$ has existence for $K^{3,\rm uq}_{\mathfrak{s}}$", so also the second possibility of clause (B)(c) of 4.32 fails; a contradiction. $\square_{4.32}$

4.34 Theorem. *1) In Theorem 4.20 we can omit the assumption (A)(b) at least if K is categorical in λ^+.*
2) $\dot I(\lambda^{++},K^{\mathfrak{u},\mathfrak{h}}) \ge \mu_{\rm unif}(\lambda^{++},2^{\lambda^+})$ <u>when</u>:

- *(A) (set theoretic) $2^\lambda < 2^{\lambda^+} < 2^{\lambda^+}$*
- *(B) (model theoretic) as in 4.20, but $\mathfrak{K}$ categorical in λ^+*
- *(C) $\mathfrak{u} = u^4_{\mathfrak{K}_\lambda}$, see Definition 4.22, $\mathfrak{h}$ is a $\mathfrak{u} - \{0,2\}$-appropriate function.*

Remark. This theorem is funny as VI§6 and in particular VI.E46-6f.13 is a shortcut, but we prove this by a detour (using VI§8) so in a sense 4.34 is less natural than 4.20; but no harm done.

Proof. 1) By part (2) and 4.24(4) recalling 4.21(4).
2) Toward contradiction, assume that the desired conclusion fails. By VI. E46-8h.1 there is an almost good λ-frame $\mathfrak{s}$ such that $\mathfrak{K}_{\mathfrak{s}} = \mathfrak{K}_\lambda$ and $\mathscr{S}^{\rm bs}_{\mathfrak{s}}(M)$ is the set of minimal $p \in \mathscr{S}^{\rm bs}_{\mathfrak{s}}(M)$.

Note that categoricity in λ^+ is used in Chapter VI to deduce the stability in λ for minimal types and the set of minimal types in $\mathscr{S}_{\mathfrak{K}}(M)$ being inevitable, but this is assumed in clause (B)(d) of the assumption of 4.20, so natural to conjecture that it is not needed, see Chapter VI.

Now using the meaning of the assumption (B)(e) of Theorem 4.20 is that "$K^{3,\rm uq}_{\mathfrak{s}}$ is not dense in $(K^{3,\rm bs}_{\mathfrak{s}}, \le_{\rm bs})$" so we can apply Theorem 4.32 to get the desired result. $\square_{4.34}$

4.35 Theorem. *1) In Theorem 4.16 we can weaken the set theoretic assumption, omitting the extra assumption (a)(β).*
2) $\dot I(\aleph_2, K^{\mathfrak{u},\mathfrak{h}}) \ge \mu_{\rm unif}(\aleph_2, 2^{\aleph_1})$ <u>when</u>:

- (a) *(set theory) $2^{\aleph_0} < 2^{\aleph_1} < 2^{\aleph_2}$*
- $(b)-(e)$ *as in 4.20*
- (f) *$\mathfrak{u} = \mathfrak{u}^3_{\mathfrak{K}_{\aleph_0}}$ from Definition 4.17 and $\mathfrak{h}$ is a $\mathfrak{u}-\{0,2\}$-appropriate function.*

<u>4.36 Discussion</u>: 1) This completes a promise from I§5. You may say that once we prove in III§3(B) that $\mathfrak{s} = \mathfrak{s}^1_{\mathfrak{K}}$ is a good $\aleph_0$-frame we do not need to deal with $\mathfrak{K}$ any more, so no need of 4.35. In addition to keeping promises this is only partially true because of the following.
2) First, arriving to $\mathfrak{s}^+$, see IV§1, we do not know that $\le_{\mathfrak{s}(+)} = \le_{\mathfrak{K}} \restriction \mathfrak{K}_{\mathfrak{s}(+)}$, because this is proved only if $\mathfrak{s}$ is good$^+$ (see IV§1). Now by looking at the definitions (and III. 600-Ex.1, equality of the various types), we know that $\mathfrak{s}$ being good$^+$ is equivalent to the symmetry property, i.e. every one sided stable amalgamation. We prove that its failure implies non-structure in 4.38, 4.39, 4.40 below.
3) Another point is that even if $\mathfrak{s}$ is weakly successful (i.e. we have existence for $K^{3,\rm uq}_{\mathfrak{s}}$), we can define ${\rm NF} = {\rm NF}_{\mathfrak{s}}$ and so we have unique non-forking amalgamation, it is not clear that this is equal to the one/two sided stable amalgamation from Chapter I.
4) Also defining $\mathfrak{s}^{+n}$ as in Chapter III we may hope not to shrink $K^{\mathfrak{s}(+n)}$, i.e. to get all the $(\aleph_0, n)$-properties (as in [Sh 87b]). If we start with $\psi \in \mathbb{L}_{\omega_1,\omega}(\mathbf{Q})$ as in [Sh 48] this seems straight, in general, this is a priori not clear, hopefully see [Sh:F888].
5) Concerning (2) above, we like to use 3.14 - 3.20 in the proof as in the proof of 6.14. If we have used $\mathfrak{u} = \mathfrak{u}^3_{\mathfrak{K}}$ from Definition 4.17, this fails, e.g. it is not self dual. We

can change $(\mathrm{FR}_2, \le_2)$ to make it symmetric but still it will fail "hereditary", so it is natural to use $\mathfrak{u}_2$ defined in 4.38 below, but then we still need $(M_\delta, N_\delta, \mathbf{I}_\delta) \in \mathrm{FR}^1_{\mathfrak{u}_2}$ to ensure N_δ is $(\mathbf{D}(M_\delta), \aleph_0)^*$-homogeneous over M_δ. This can be done by using the game version of the coding property. This is fine but was not our "main road" so rather we use the theorem on $\mathfrak{u}_3$ but use $\mathfrak{u}_5$ to apply §3.

A price of using §3 is having to use fake equality. Also together with symmetry, we deal with lifting free $(\alpha, 0)$-rectangles.
6) To complete the proof of 4.35, by 4.40 it suffices to prove the uniqueness of two-sided stable amalgamation. We use §8 and toward this we define WNF_*, prove that it is a weak non-forking relation of $\mathfrak{K}_{\aleph_0}$ respecting $\mathfrak{s}$, using the "lifting" from §5. Then we can apply §8.
7) A drawback of 4.35 as well as 4.16 and III§3(B) is that we restrict ourselves to a countable $\mathbf{D}$. Now in Chapter I this is justified as it is proved that for some increasing continuous sequence $\langle \mathbf{D}_\alpha : \alpha < \omega_1 \rangle$ with each $\mathbf{D}_\alpha$ countable, $\mathbf{D} = \cup\{\mathbf{D}_\alpha : \alpha < \omega_1\}$, i.e. for every $M \in K_{\aleph_0}$, the sequence $(\mathbf{D}_\alpha(M) : \alpha < \omega_1\rangle$ is an increasing sequence of sets of types with union $\mathbf{D}(M)$. However, from the positive results on every $\mathbf{D}_\alpha$ we can deduce positive results on $\mathbf{D}$. See, hopefully, in [Sh:F888].

4.37 Remark. 1) Assumption (d) of 4.16 gives: usually $(M, N, \mathbf{I}) \in \mathrm{FR}^1_{\mathfrak{u}}$ has non-uniqueness, i.e. when $\mathbf{I} = ({}^{\omega>}N)\backslash({}^{\omega>}M)$. We like to work as in 4.32.
2) So as indirectly there we would like to use 3.24; for this we need the vertical uq-invariant whereas we naturally get failure of the semi uq-invariant coding property. So we would like to quote 3.20 but this requires $\mathfrak{u}$ to be self dual.
3) Hence use also a relative of $\mathfrak{u}$ from 4.41, for it we prove the implication and from this deduce what we need for the old.
4) Our problem is to prove that $\mathfrak{s} = \mathfrak{s}_{\aleph_0}$ is good$^+$, equivalently prove the symmetry property, this is done in Claim 4.40. It is natural to apply 3.13 - 3.24.
5) The proof of 4.35 will come later.

4.38 Definition. In 4.35 we define $\mathfrak{u}_5 = \mathfrak{u}^5_{\mathfrak{K}_{\aleph_0}}$ as follows (ℓ is $1,2$)

- (a) $\partial = \partial_{\mathfrak{u}} = \aleph_1$
- (b) $\mathfrak{K}_{\mathfrak{u}} = \mathfrak{K}_{\aleph_0}$ more pedantically $\mathfrak{K}_{\mathfrak{u}} = \mathfrak{K}'_{\aleph_0}$
- $(c)_1$ $\mathrm{FR}^{\mathfrak{u}}_1$ is the class of triples $(M, N, \mathbf{I})$ such that $\mathbf{I} \subseteq {}^{\omega>}N\backslash{}^{\omega>}M$
- $(c)_2$ $(M_1, N_1, \mathbf{I}_1) \le_1 (M_1, N_2, \mathbf{I}_2)$ iff both are from $\mathrm{FR}^{\mathfrak{u}}_1, M_1 \le_{\mathfrak{K}} M_2, N_1 \le_{\mathfrak{K}} N_2$ and $\bar c \in \mathbf{I} \Rightarrow$ $\mathrm{gtp}(\bar c, M_2, N_2)$ is the stationarization of $\mathrm{gtp}(\bar c, M_1, N_1)$
- (d) $\mathrm{FR}_2 =$ FR_1 and $\le^2_{\mathfrak{u}} = \le^1_{\mathfrak{u}}$.

Now we have to repeat various things.

4.39 Claim. *1) $\mathfrak{u}_5$ is a nice construction framework.*
2) For almost$_2$ every triples $(\bar M, \bar{\mathbf{J}}, \mathbf{f}) \in K^{\rm qt}_{\mathfrak{u}_5}$ the model $M_\partial = M_{\lambda^+}$ belongs to K_{λ^+} and is saturated.
3) $\mathfrak{u}_5$ has fake equality $=_\tau$ and is monotonic, see Definition (1.13(1)), and weakly hereditary for the fake equality $=_\tau$, see Definition 3.17(5) and interpolative (see Definition 3.21).

Proof. Should be clear (e.g. part (2) as in 6.2). $\square_{4.39}$

4.40 Claim. *$\dot I(\lambda^{++}, \mathfrak{K}) \ge \mu_{\rm unif}(\aleph_2, 2^{\aleph_0})$ and moreover $\dot I(\aleph_2, \mathfrak{K}(\aleph_1 - saturated)) \ge \mu_{\rm unif}(\aleph_2, 2^{\aleph_2})$ <u>when</u>:*

- $\circledast$ *$(a)(\alpha), (b), (c), (e)$ from 4.16 and*
  - *$(d)''(\alpha)$ $\mathfrak{K}$ fails the symmetry property <u>or</u>*
  - *(β) $\mathfrak{K}$ fails the lifting property, see Definition 4.41 below.*

4.41 Definition. We define a 4-place relation WNF_* on $K_{\aleph_0}$ as follows:
$\mathrm{WNF}_*(M_0, M_1, M_2, M_3)$ <u>when</u>

- (a) $M_\ell \in K_{\aleph_0}$ for $\ell \le 3$
- (b) $M_0 \le_{\mathfrak{K}} M_\ell \le_{\mathfrak{K}} M_3$ for $\ell = 1, 2$
- (c) for $\ell = 1, 2$ if $\bar a \in {}^{\omega>}(M_\ell)$ then $\mathrm{gtp}(\bar a, M_{3-\ell}, M_3)$ is the stationarization of $\mathrm{gtp}(\bar a, M_0, M_3) = \mathrm{gtp}(\bar a, M_0, M_\ell)$.

4.42 Definition. We say that $(\mathfrak{K}, \mathrm{WNF}_*)$ has the lifting property when WNF_* satisfies clause (g) of Definition 7.18, see the proof of 7.18, i.e. if $\mathrm{WNF}_*(M_0, N_0, M_1, N_1)$ and $\alpha < \lambda^+$ and $\langle M_{0,i} : i \le \alpha\rangle$ is $\le_{\mathfrak{s}}$-increasing continuous, $M_{0,0} = M_0$ and $N_0 \le_{\mathfrak{K}_\lambda}, M_{0,\alpha}$ <u>then</u> we can find a $\le_{\mathfrak{K}_\lambda}$-increasing continuous sequence $\langle M_{1,i} : i \le \alpha + 1\rangle$ such that $M_{1,0} = M_1, N_1 \le_{\mathfrak{K}} M_{1,\alpha+1}$ and for each $i < \alpha$ we have $\mathrm{WNF}_*(M_{0,i}, M_{0,i+1}, M_{1,i}, M_{1,i+1})$ for $i < \alpha$.

Proof of 4.40. We start as in the proof of 4.16, choosing the good $\aleph_0$-frame $\mathfrak{s} = \mathfrak{s}_{\aleph_0}$ and define $\mathfrak{u} = \mathfrak{u}^3_{\mathfrak{K}}$ as there, (except having the fake inequality which causes no problem), so it is a nice construction framework by 4.18(1) and for almost$_2$ all triples $(\bar M, \bar{\mathbf{J}}, \mathbf{f}) \in K^{\rm qt}_{\mathfrak{u}}$ the model $M \in \mathfrak{K}_{\aleph_1}$ is saturated (by 4.18(2)).

Now Theorem 3.24 gives the right conclusion, so to suffice to verify its assumptions. Of course, $\mathfrak{u}$ is as required in Hypothesis 3.1.

Clause (a) there means $2^{\aleph_0} < 2^{\aleph_0} < 2^{\aleph_2}$ (as $\partial_{\mathfrak{u}} = \aleph_1$ and we choose $\theta = \aleph_0$), which holds by clause $(a)(\alpha)$ of the present claim.

Clause (c) there says that for $\{0,2\}$-almost every $(\bar{M}, \bar{\mathbf{J}}, \mathbf{f}) \in K^{\rm qt}_{\mathfrak{u}}$ the model $M_\partial \in K_{\aleph_2}$ is $K_{\mathfrak{u}}$-model homogeneous; this holds and can be proved as in 6.2.

We are left with clause (b), i.e. we have to prove that some $(M, N, \mathbf{I}) \in {\rm FR}^{\mathfrak{u}}_1$ has the vertical uq-invariant coding property, see Definition 3.10. Choose $(M, N, \mathbf{I}) \in {\rm FR}^{\mathfrak{u}}_1$ such that $|\mathbf{I}| > 1$, hence N is $(\mathbf{D}(M), N)^*$-homogeneous and $\mathbf{I} = ({}^{\omega >}N) \backslash ({}^{\omega >}M)$ and we shall prove that it has the vertical uq-invariant coding, so assume

- $(*)$ $\mathbf{d}_0$ is a $\mathfrak{u}$-free $(\alpha_{\mathbf{d}}, 0)$-rectangle satisfying $M \leq_{\mathfrak{s}} M' = M^{\mathbf{d}}_{0,0}$ and $M^{\mathbf{d}}_{\alpha(\mathbf{d}),0} \cap N = M$.

We have to find $\mathbf{d}$ as required in Definition 3.10.

Note that by the choice of $(M, N, \mathbf{I})$, and the assumption "$\mathfrak{K}$ fails the symmetry property" we can find (M_*, N_*) and then $\bar{c}$

$(*)_2$ (a) $M \leq_{\mathfrak{s}} M_* \leq_{\mathfrak{s}} N_*$ and $N \leq_{\mathfrak{s}} N_*$

(b) M, N, M_*, N_* is in one-sided stable amalgamation, i.e. if $\bar{b} \in {}^{\omega >}N$ then ${\rm gtp}(\bar{b}, M_*, N_*)$ is the stationarization of ${\rm gtp}(\bar{b}, M, N)$

(c) M, M_*, N, N_* is not in one sided stable amalgamation, so

$(c)^+$ $\bar{c} \in {}^{\omega >}(M_*)$ and ${\rm gtp}(\bar{c}, N, N_*)$ is not the stationarization of ${\rm gtp}(\bar{c}, M, M_*)$.

We like to apply the semi version, i.e. Definition 3.14 and Claim 3.20. There are technical difficulties so we apply it to $\mathfrak{u}_5$, see 4.38, 4.39 above and in the end increase the models to have the triples in ${\rm FR}^1_{\mathfrak{u}}$ and use 3.23 instead of 3.20, so all should be clear.

Alternatively, works only with $\mathfrak{u}_5$ but use the game version of the coding theorem. $\square_{4.40}$

4.43 Claim. *1) If ($\mathfrak{K}$,*WNF$_*$*) has lifting, see Definition 4.41, then* WNF$_*$ *is a weak non-forking relationg of $\mathfrak{K}_{\aleph_0}$ respecting $\mathfrak{s}$ with disjointness (7.18(3)).*
2) WNF$_*$ *is a pseudo non-forking relation of $\mathfrak{K}_{\aleph_0}$ respecting $\mathfrak{s}$ meaning clauses (a)-(f) with disjointness, see the proof or see Definition 7.18(4),(3).*
3) If $\mathfrak{K}_{\aleph_0}$ satisfies symmetry then in clause (c) of Definition 4.41, it is enough if it holds for one ℓ.

Proof. 1) We should check all the clauses of Definition 7.18, so see III. 600-nf.0X or the proof of 7.17(1).

Clause (a): WNF_* is a 4-place relation on $\mathfrak{K}_{\aleph_0}$.
[Why? By Definition 4.41, in particular clause (a).]

Clause (b): $\mathrm{WNF}_*(M_0, M_1, M_2, M_3)$ implies $M_0 \le_{\mathfrak{K}} M_\ell \le_{\mathfrak{K}} M_3$ for $\ell = 1, 2$ and is preserved by isomorphisms.
[Why? By Definition 4.41, in particular clause (c).]

Clause (c): Monotonicity
[Why? By properties of gtp, see I.88r-5.13(1).]

Clause (d): Symmetry
[Why? Read Definition 4.41.]

Clause (e): Long Transitivity
As in the proof of I.88r-5.19.

Clause (f): Existence
This is proved in I.88r-5.22.

Clause (g): Lifting, see Definition 4.41.
This holds by an assumption.
WNF_* respects $\mathfrak{s}$ and has disjointness.
Clear by the definition (in particular of gtp).
2) The proof included in the proof of part (1).
3) Should be clear. $\square_{4.43}$

Proof of 4.35. Let $\lambda = \aleph_0$ and toward contradiction assume that $\dot{I}(\lambda^{++}, \mathfrak{K}(\lambda^+\text{-saturated})) < \mu_{\mathrm{unif}}(\lambda^{++}, 2^\lambda)$.

As in the proof of 4.16, by III.600-Ex.1 $\mathfrak{s} := \mathfrak{s}_{\aleph_0}$ is a good λ-frame categorical in λ. By Theorem 4.32, recalling our assumption toward contradiction, $K^{3,\mathrm{uq}}_{\mathfrak{s}}$ is dense in $K^{3,\mathrm{bs}}_{\mathfrak{s}}$ hence $\mathfrak{s}$ has existence for $K^{3,\mathrm{uq}}_{\mathfrak{s}}$; i.e. is weakly successful, but we shall not use this.

By 4.40 we know that $\mathfrak{K}$ has the symmetry property, hence the two-sided stable amalgamation fails uniqueness. Also by 4.40 we know that it has the lifting property, so by 4.43, 4.40 we know that WNF_* is a weak non-forking relation on $\mathfrak{K}_{\aleph_0}$ which respects $\mathfrak{s}$, so Hypothesis 8.1 holds.

Let $\mathfrak{u}$ be defined as in 8.3 (for our given $\mathfrak{s}$ and WNF_*). Now we try to apply Theorem 8.19. Its conclusion fails by our assumption toward contradiction and clause (a),(b),(c) there holds. So clause (b) there fails so by 8.12(1). So we can conclude that we have uniqueness for WNF_* by 8.12(2) clearly $\mathfrak{s}$ has existence for $K^{3,\mathrm{bs}}_{\mathfrak{s}}$, i.e. is weakly successful.

So $\mathfrak{s}^+$ is a well defined good λ^+-frame, see Chapter IV. By III§8, Chapter IV and our assumption toward contradiction, we know that $\mathfrak{s}$ is successful. Now if $\mathfrak{s}$ is

not good$^+$ then $\mathfrak{K}$ fails the symmetry property hence by 4.40 we get contradiction, so ncessarily $\mathfrak{s}$ is good$^+$ hence we have $\leq_{\mathfrak{s}(+)} = \leq_{\mathfrak{s}} \restriction K_{\mathfrak{s}(+)}$. This proves that the saturated $M \in \mathfrak{K}_{\lambda^+}$ is super limit (see Chapter IV also this is I. 88r-5.24). $\square_{4.35}$

§5 On almost good λ-frames

Accepting "WDmId(∂) is not ∂^+-saturated" where $\partial = \lambda^+$ we have accomplished in §4 the applications we promised. Otherwise for III§5 we have to prove for a good λ-frame $\mathfrak{s}$ that among the triples $(M, N, a) \in K^{3,\mathrm{bs}}_{\mathfrak{s}}$, the ones with uniqueness are not dense, as otherwise non-structure in λ^{++} follows. Toward this (in this section) we have to do some (positive structure side) work which may be of self interest. Now in the case we get $\mathfrak{s}$ in III§3 starting from [Sh 576] or better from VI§8, but with $\lambda_{\mathfrak{s}} = \lambda$ rather than $\lambda_{\mathfrak{s}} = \lambda^+$, we have to start with an almost good λ-frames $\mathfrak{s}$ rather than with good λ-frames. However, there is a price: for eliminating the non-∂^+-saturation of the weak diamond ideal and for using the "<u>almost</u> λ-good" version, we have to work more.

First, we shall not directly try to prove density of uniqueness triples $(M, N, \mathbf{J})$ but just the density of poor relatives like $K^{3,\mathrm{up}}_{\mathfrak{s}}$.

Second, we have to prove some positive results, particularly in the almost good λ-frame case. This is done here in §5 and more is done in §7 assuming existence for $K^{3,\mathrm{up}}_{\mathfrak{s},\lambda^+}$ justified by the non-structure result in §6 and the complimentary full non-structure result is proved in §8.

5.1 Hypothesis. $\mathfrak{s}$ is an almost good λ-frame (usually categorical in λ) and for transparency $\mathfrak{s}$ has disjointness, see Definitions 5.2, 5.5 below; the disjointness is justified in the Discussion 5.6 and not used in 5.21 - 5.25 which in fact prove it and let $\partial = \lambda^+$.

5.2 Definition. "$\mathfrak{s}$ is an almost good λ-frame" is defined as in III. 600-1.1 except that we weaken (E)(c) to (E)(c)$^-$ and strengthen (D)(d) to (D)(d)$^+$ where (recall $\mathbf{tp}_{\mathfrak{s}} = \mathbf{tp}_{\mathfrak{K}_{\mathfrak{s}}}$):

<u>Ax(E)(c)$^-$</u>: the local character

<u>if</u> $\langle M_i : i \leq \delta + 1\rangle$ is $\leq_{\mathfrak{s}}$-increasing continuous and the set $\{i < \delta : N_i <^*_{\mathfrak{s}} N_{i+1}$, i.e. N_{i+1} is universal over $N_i\}$ is unbounded in δ <u>then</u> for some $a \in M_{\delta+1}$ the type $\mathbf{tp}_{\mathfrak{s}}(a, M_\delta, M_{\delta+1})$ belongs to $\mathscr{S}^{\mathrm{bs}}_{\mathfrak{s}}(M_\delta)$ and does not fork over M_i for some $i < \delta$

Ax$(D)(d)^+$ if $M \in K_{\mathfrak{s}}$ then $\mathscr{S}_{\mathfrak{K}_{\mathfrak{s}}}(M)$ has cardinality $\leq \lambda$
(for good λ-frame this holds by III. 600-4a.1).

As in Chapter III

5.3 Definition. 1) $K_{\mathfrak{s}}^{3,\mathrm{bs}}$ is the class of triples (M, N, a) such that $M \le_{\mathfrak{s}} N$ and $a \in N \backslash M$.
2) $\le_{\mathrm{bs}} = \le_{\mathfrak{s}}^{\mathrm{bs}}$ is the following two-place relation (really partial order) on $K_{\mathfrak{s}}^{3,\mathrm{bs}}$. We let $(M_1, N_1, a_1) \le_{\mathrm{bs}} (M_2, N_2, a_2)$ iff $a_1 = a_2, M_1 \le_{\mathfrak{s}} M_2, N_1 \le_{\mathfrak{s}} N_2$ and $\mathbf{tp}_{\mathfrak{s}}(a_1, N_1, N_2)$ does not fork over M_1.

5.4 Claim. *1) $K_{\mathfrak{s}}^{3,\mathrm{bs}}$ and $\le_{\mathrm{bs}}$ are preserved by isomorphisms.*
2) $\le_{\mathrm{bs}}$ is a partial order on $K_{\mathfrak{s}}^{3,\mathrm{bs}}$.
3) If $\langle (M_\alpha, N_\alpha, a) : \alpha < \delta \rangle$ is $\le_{\mathrm{bs}}$-increasing and $\delta < \lambda^+$ is a limit ordinal and $M_\delta := \cup\{M_\alpha : \alpha < \delta\}, N_\delta := \cup\{N_\alpha : \alpha < \delta\}$ <u>then</u> $\alpha < \delta \Rightarrow (M_\alpha, N_\alpha, a) \le_{\mathrm{bs}} (M_\delta, N_\delta, a) \in K_{\mathfrak{s}}^{3,\mathrm{bs}}$ (using Ax(E)(h)).

Proof. Easy. $\square_{5.4}$

5.5 Definition. We say $\mathfrak{s}$ has disjointness or is disjoint <u>when</u>:

- (a) strengthen Ax(C), i.e. $\mathfrak{K}_{\mathfrak{s}}$ has disjoint amalgamation which means that: if $M_0 \le_{\mathfrak{s}} M_\ell$ for $\ell = 1, 2$ and $M_1 \cap M_2 = M_0$ <u>then</u> for some $M_{\mathfrak{s}} \in K_{\mathfrak{s}}$ we have $M_\ell \le_{\mathfrak{s}} M_{\mathfrak{s}}$ for $\ell = 0, 1, 2$
- (b) strengthen Ax(E)(i) by disjointness: if above we assume in addition that $(M_0, M_\ell, a_\ell) \in K_{\mathfrak{s}}^{3,\mathrm{bs}}$ for $\ell = 1, 2$ <u>then</u> we can add $(M_0, M_\ell, a_\ell) \le_{\mathrm{bs}} (M_{3-\ell}, M_{\mathfrak{s}}, a_\ell)$ for $\ell = 1, 2$.

<u>5.6 Discussion</u>: How "expensive" is the (assumption of) disjoint amalgamation (in Ax(C) and Ax(E)(i))?
1) We can "get it for free" by using $\mathfrak{K}'$ and $\mathfrak{s}'$, see Definition 1.10 and 4.25(2) so we assume it.
2) Alternatively we can prove it assuming categoricity in λ (see 5.23 which relies on 5.22).
3) So usually we shall ignore this point.

5.7 Exercise. There is a good λ-frame without disjoint amalgamation.

[Hint: Let

$\circledast_1$ (a) $\tau = \{F\}, F$ a unary function
(b) ψ the first order sentence $(\forall x, y)[F(x) \neq x \wedge F(y) \neq y \rightarrow F(x) = F(y)] \wedge ((\forall x)[F(F(x)) = F(x)]$

(c) $K = \{M : M \text{ is a } \tau\text{-model of } \psi\}$, so $M \in K \Rightarrow |\varphi(M)| \le 1$ where we let $\varphi(x) = (\exists y)(F(y) = x \wedge y \ne x)$

(d) $M \le_{\mathfrak{K}} N'$ iff $M \subseteq N$ are from K and $\varphi(M) = \varphi(N) \cap M$.

Now note

$(*)_1$ $\mathfrak{K} := (K, \le_{\mathfrak{K}})$ is an a.e.c. with $\mathrm{LS}(\mathfrak{K}) = \aleph_0$

$(*)_2$ $\mathfrak{K}$ has amalgamation.

[Why? If $M_0 \le_{\mathfrak{K}} M_\ell$ for $\ell = 1, 2$, then separate the proof to three cases: the first when $\varphi(M_0) = \varphi(M_1) = \varphi(M_2) = \emptyset$ the second when $|\varphi(M_1)| + |\varphi(M_2)| = 1$ and $\varphi(M_0) = \emptyset$; in the third case $\varphi(M_0) = \varphi(M_1) = \varphi(M_2)$ is a singleton.]

So

$(*)_3$ $\mathfrak{K}_\lambda = (K_\lambda, \le_{\mathfrak{K}} \restriction K_\lambda)$.

Now we define $\mathfrak{s}$ by letting

$(*)_4$ (a) $\mathfrak{K}_{\mathfrak{s}} = \mathfrak{K}_\lambda$

(b) $K^{3,\mathrm{bs}}_{\mathfrak{s}} := \{(M, N, a) : M \le_{\mathfrak{K}_\lambda} N, a \in N \backslash M \backslash \varphi(N)\}$

(c) for $M_1 \le_{\mathfrak{K}} M_2 \le_{\mathfrak{K}} M_3$ and $a \in M_3$ we say $\mathbf{tp}_{\mathfrak{K}}(a, M_2, M_3)$ does not fork over M_1 iff $a \notin M_2$ & $F^{M_3}(a) \notin M_2 \backslash M_1$.

Lastly

$(*)_5$ $\mathfrak{s}$ is a good λ-frame.

[Why? Check. E.g.

<u>Ax(D)(c)</u>: Density

So assume $M <_{\mathfrak{s}} N$ now if there is $a \in N \backslash M \backslash \varphi(M)$ then a $\mathbf{tp}(a, M, N) \in \mathscr{S}^{\mathrm{bs}}_{\mathfrak{s}}(M)$ so a is as required. Otherwise, as $M \ne M$ necessary $\varphi(N)$ is non-empty and $\subseteq N \backslash M$, let it be $\{b\}$. By the definition of φ there is $a \in N$ such that $F^N(a) = b \wedge a \ne b$ so necessarily $a \notin M$ and is as required.]

<u>Ax(E)(e)</u>: Uniqueness

The point is that:

$(*)_6$ if $\varphi(M) = \emptyset$ then $\mathscr{S}^{\mathrm{bs}}_{\mathfrak{s}}(M)$ contains just two types p_1, p_2 such that if $p_\ell = \mathbf{tp}(a, M, N) \Rightarrow$ then $\ell = 1 \Rightarrow F^N(a) = a \in N \backslash M$ and $\ell = 2 \Rightarrow F^N(a) \in N \backslash M \backslash \{a\}$

$(*)_7$ if $\varphi(M) = \{b\}$ then $\mathscr{S}^{\mathrm{bs}}_{\mathfrak{s}}(M)$ contains just two types p_1, p_2 such that p_1 is as above and $p_2 = \mathbf{tp}(a, M, N) \Rightarrow F^N(a) = b$

$(*)_8$ there are $M_0 \le_{\mathfrak{K}} M_1 = M_2$ such that $\varphi(M_1) \ne \emptyset = \varphi(M_0)$ and let $\varphi(M_\ell) = \{b\}$ for $\ell = 1,2$ so $b \in M_\ell \backslash M_0$. So we cannot disjointly amalgamate M_1, M_2 over M_0.

[Why? Think.]
So we are done with Example 5.7.]

Recalling III.600-0.21, III.600-0.22:

5.8 Claim. *1) For $\kappa = \mathrm{cf}(\kappa) \le \lambda$*

- (a) *there is a (λ,κ)-brimmed $M \in \mathfrak{K}_{\mathfrak{s}}$, in fact (λ,κ)-brimmed over M_0 for any pregiven $M_0 \in \mathfrak{K}_{\mathfrak{s}}$*
- (b) *M is unique up to isomorphism over M_0 (but we fix κ)*
- (c) *if $M \in \mathfrak{K}_{\mathfrak{s}}$ is (λ,κ)-brimmed over M_0 <u>then</u> it is $\le_{\mathfrak{s}}$-universal over M_0.*

2) So the superlimit $M \in \mathfrak{K}_{\mathfrak{s}}$ is (λ,κ)-brimmed for every $\kappa \le \mathrm{cf}(\kappa) \le \lambda$ hence is brimmed.
3) If $\kappa = \mathrm{cf}(\kappa) \le \lambda$ and $M_1 \le_{\mathfrak{s}} M_2$ are both (λ,κ)-brimmed and $\Gamma \subseteq \mathscr{S}^{\mathrm{bs}}_{\mathfrak{s}}(M_2)$ has cardinality $< \kappa$ and every $p \in \Gamma$ does not fork over M_1 <u>then</u> there is an isomorphism f from M_2 onto M_1 such that $p \in \Gamma \Rightarrow f(p) = p \restriction M_1$.

Proof. 1) By Definition III.600-0.21 and Claim III.600-0.22 because $\mathfrak{K}_\lambda$ has amalgamation, the JEP recalling III.600-1.1 and has no $<_{\mathfrak{K}_\lambda}$-maximal member (having a superlimit model).
2) By the definition of being brimmed and "superlimit in $\mathfrak{K}_{\mathfrak{s}}$" which exists by "$\mathfrak{s}$ is almost good λ-frame".
3) Exactly as in the proof of IV.705-stg.9. $\square_{5.8}$

5.9 Remark. 1) It seems that there is no great harm in weakening (E)(h) to (E)(h)$^-$ as in (E)(c)$^-$, but also no urgent need, where:.

Ax(E)(h)$^-$: assume $\langle M_i : i \le \delta\rangle$ is $\le_{\mathfrak{s}}$-increasing continuous and $\delta = \sup\{i : M_{i+1}$ is $\le_{\mathfrak{s}}$-universal over $M_i\}$. If $p \in \mathscr{S}_{\mathfrak{K}_{\mathfrak{s}}}(M_\delta)$ and $i < \delta \Rightarrow p \restriction M_i \in \mathscr{S}^{\mathrm{bs}}_{\mathfrak{s}}(M_i)$ <u>then</u> $p \in \mathscr{S}^{\mathrm{bs}}_{\mathfrak{s}}(M_\delta)$.
2) That is, if we weaken Ax(E)(h) then we are drawn to further problems. After defining $\mathfrak{u}$, does $\le_\ell$-increasing sequence $\langle (M_i, N_i, \mathbf{J}_i) : i < \delta\rangle$ has the union as a $\le_\ell$-upper bound? If M_{i+1} is universal over M_i for $i < \delta$ this is O.K., but using triangles in the limit we have a problem; see part (4) below.
3) Why "no urgent need"? The case which draws us to consider Ax(E)(c)$^-$ is VI§8, i.e. by the $\mathfrak{s}$ derived there satisfies Ax(E)(h). So we may deal with it elsewhere,

[Sh:F841].
4) When we deal with $\mathfrak{u}$ derived from such $\mathfrak{s}$, i.e. as in part (1) we may demand:

- (A) First, dealing with $\mathfrak{u}$-free rectangles and triangles we add
  - (a) $\mathbf{I}^{\mathbf{d}}_{i,j} = \emptyset$ when j is a limit ordinal
  - (b) $\mathbf{J}^{\mathbf{d}}_{i,j} = \emptyset$ when i is a limit ordinal
  - (c) it is everywhere universal (each $M_{i+1,j+j}$ is $\leq_{\mathfrak{s}}$-universal over $M^{\mathbf{d}}_{i+1,j} \cup M^{\mathbf{d}}_{i,j+1}$ (or at least each)
- (B) similarly with $K^{\rm qt}_{\mathfrak{s}}$, i.e. defining $(\bar{M}, \bar{\mathbf{J}}, \mathbf{f}) \in K^{\rm qt}_{\mathfrak{s}}$ we add the demands
  - (a) $\mathbf{J}_\delta = \emptyset$ for limit δ
  - (b) if $\delta \in S \cap E$, then $\mathbf{d}$, the $(\mathfrak{u}_{\mathfrak{s}})$-free $(\mathbf{f}(\delta), 0)$-rectangle $(\langle M_{\delta+i} : i \leq \mathbf{f}(\delta)\rangle, \langle \mathbf{J}_{\delta+i} : i < \mathbf{f}(\delta)\rangle)$ is strongly full (defined as below) and so for any $\theta \leq \lambda$, if $\lambda | j$ and $i < j \leq \mathbf{f}(\delta)$ and $\mathrm{cf}(\delta) = \theta$ then $M_{\delta,j}$ is (λ, θ)-brimmed over $M_{\delta+i}$.
- (C) similarly for $\leq^{\rm at}_{\mathfrak{u}_{\mathfrak{s}}}, \leq^{\rm qr}_{\mathfrak{u}_{\mathfrak{s}}}$.

5.10 Definition. 1) We define $\mathfrak{u} = \mathfrak{u}_{\mathfrak{s}} = \mathfrak{u}^1_{\mathfrak{s}}$ as in Definition 4.29, it is denoted by $\mathfrak{u}$ for this section so may be omitted and we may write $\mathfrak{s}$-free instead of $\mathfrak{u}_{\mathfrak{s}}$-free.
2) We say $(M, N, \mathbf{I}) \in \mathrm{FR}_\ell$ realizes p <u>when</u> $\mathbf{tp}_{\mathfrak{s}}(a_{\mathbf{I}}, M, N) = p$, recalling $\mathbf{I} = \{a_{\mathbf{I}}\}$.

5.11 Claim. *1) $\mathfrak{u}_{\mathfrak{s}}$ is a nice construction framework which is self-dual.*
2) Also $\mathfrak{u}_{\mathfrak{s}}$ is monotonic and hereditary and interpolative.

5.12 Remark. 1) Here we use "$\mathfrak{s}$ has disjointness" proved in 5.23.
2) Even without 5.23, if $\mathfrak{s} = \mathfrak{s}'_1$ for some almost good λ-frame $\mathfrak{s}_1$ then $\mathfrak{s}$ has disjointness.
3) Mostly it does not matter if we use $\mathfrak{u}^1_{\mathfrak{s}}, \mathfrak{s}'$ from 4.25 (see 6.18) but in proving 6.13(1), the use of $\mathfrak{u}^1_{\mathfrak{s}'}$ is preferable; alternatively in defining nice construction framework we waive the disjointness, which is a cumbersome but not serious change.

Proof. As in 4.30 except disjointness which holds by Hypothesis 5.2 and is justified by 5.6 above, (or see 5.23 below). $\square_{5.11}$

5.13 Remark. Because we assume on $\mathfrak{s}$ only that it is an almost good λ-frame we have to be more careful as (E)(c) may fail, in particular in proving brimmness in

triangles of the right kind. I.e. for a $\mathfrak{u}$-free $(\bar\alpha,\beta)$- triangle $\mathbf{d}$, we need that in the "vertical sequence", $\langle M^{\mathbf{d}}_{i,\beta} : i \le \alpha_\beta\rangle$ the highest model $M^{\mathbf{d}}_{\alpha_\beta,\beta}$ is brimmed over the lowest $M^{\mathbf{d}}_{0,\beta}$. This motivates the following.

5.14 Definition. 1) We say $\bar M$ is a (Γ,δ)-correct sequence <u>when</u>: $\Gamma \subseteq \mathscr{S}^{\rm bs}_{\mathfrak{s}}(M_\delta)$, the sequence $\bar M = \langle M_\alpha : \alpha \le \alpha(*)\rangle$ is $\le_{\mathfrak{s}}$-increasing continuous, $\delta \le \alpha(*)$ and: <u>if</u> $N \in K_{\mathfrak{s}}$ satisfies $M_\delta <_{\mathfrak{s}} N$ <u>then</u> for some $c \in N \backslash M_\delta$ and $\alpha < \delta$ the type $\mathbf{tp}_{\mathfrak{s}}(c, M_\delta, N)$ does not fork over M_α and belongs to Γ.
2) We omit δ when this holds for every limit $\delta \le \alpha(*)$.
3) We say Γ is M-inevitable when $\Gamma \subseteq \mathscr{S}^{\rm bs}_{\mathfrak{s}}(M)$ and: if $M <_{\mathfrak{s}} N$ <u>then</u> some $p \in \mathscr{S}^{\rm bs}_{\mathfrak{s}}(M) \cap \Gamma$ is realized in N.
4) Using a function $\mathscr{S}^*$ instead of Γ we mean we use $\Gamma = \mathscr{S}^*(M_\delta)$.
5) We may omit Γ (and write δ-correct) above <u>when</u> Γ is $\mathscr{S}^{\rm bs}_{\mathfrak{s}}$.
6) For $\bar M = \langle M_\alpha : \alpha \le \alpha(*)\rangle$ let $\text{correct}_\Gamma(\bar M) = \{\delta \le \alpha(*) : \bar M$ is (δ,Γ)-correct so δ is a limit ordinal$\}$ and we may omit Γ if $\Gamma = \mathscr{S}^{\rm bs}_{\mathfrak{s}}(M_{\alpha(*)})$.
7) If $\mathbf{d}$ is a $\mathfrak{u}_{\mathfrak{s}}$-free $(\bar\alpha,\beta)$-triangle let $\Gamma_{\mathbf{d}} = \{p \in \mathscr{S}^{\rm bs}_{\mathfrak{s}}(M_{\alpha_\beta,\beta}) : p$ does not fork over $M_{i,j}$ for some $j < \beta, i < \alpha_j\}$.

5.15 Definition. 1) We say that $\mathbf{d}$ is a brimmed (or universal) $\mathfrak{u}_{\mathfrak{s}}$-free or $\mathfrak{s}$-free triangle <u>when</u>:

- (a) $\mathbf{d}$ is a $\mathfrak{u}_{\mathfrak{s}}$-free triangle
- (b) if $i < \alpha_j(\mathbf{d})$ and $j < \beta(\mathbf{d})$ <u>then</u> $M^{\mathbf{d}}_{i+1,j+1}$ is brimmed (or universal) over $M^{\mathbf{d}}_{i+1,j}$.

2) We say strictly brimmed (universal) <u>when</u> also $M^{\mathbf{d}}_{i+1,j+1}$ is brimmed/universal over $M^{\mathbf{d}}_{i+1,j} \cup M^{\mathbf{d}}_{i,j+1}$ when $j < \beta, i < \alpha_j(\mathbf{d})$.
2A) We say that $\mathbf{d}$ is a weakly brimmed (or weakly universal) $\mathfrak{u}_{\mathfrak{s}}$-free or $\mathfrak{s}$-free triangle <u>when</u>:

- (a) $\mathbf{d}$ is a $\mathfrak{u}_{\mathfrak{s}}$-free triangle
- (b) if $j_1 < \beta_{\mathbf{d}}, i_1 < \alpha_j(\mathbf{d})$ then we can find a pair (i_2,j_2) such that $j_1 \le j_2 < \beta_{\mathbf{d}}, i_1 \le i_2 < \alpha_j(\mathbf{d})$ and $M^{\mathbf{d}}_{i_2+1,j_2+1}$ is brimmed (or is $\le_{\mathfrak{s}}$-universal) over $M^{\mathbf{d}}_{i_2,j_2}$ or just over $M^{\mathbf{d}}_{i_1,j_1}$.

2B) We say that $\mathbf{d}$ is a weakly brimmed (weakly universal) $\mathfrak{u}_{\mathfrak{s}}$-free rectangle <u>when</u> it and its dual (see Definition 1.7(3)) are weakly brimmed $\mathfrak{u}_{\mathfrak{s}}$-free triangles. Similarly for brimmed, strictly brimmed, universal, strictly universal.
3) We say that a $\mathfrak{u}_{\mathfrak{s}}$-free triangle $\mathbf{d}$ is full <u>when</u>:

(a) $\lambda_{\mathfrak{s}}$ divides $\alpha_{\beta(\mathbf{d})}(\mathbf{d})$ and $\beta(\mathbf{d})$ is a limit ordinal and $\bar{\alpha}$ is continuous or just $\alpha_{\beta(\mathbf{d})}(\mathbf{d}) = \cup\{\alpha_j(\mathbf{d}) : j < \beta\}$

(b) if $i < \alpha_j(\mathbf{d}), j < \beta := \beta(\mathbf{d})$ and $p \in \mathscr{S}^{\rm bs}_{\mathfrak{s}}(M_{i,j})$ <u>then</u>:

  $(*)$ the following subset $S^{\mathbf{d}}_p$ of $\alpha_\beta(\mathbf{d})$ has order type $\geq \lambda_{\mathfrak{s}}$ where $S^{\mathbf{d}}_p := \{i_1 < \alpha_\beta(\mathbf{d})$: for some $j_1 \in (j,\beta)$ we have $i \leq i_1 < \alpha_{j_1}(\mathbf{d})$ and for some $c \in \mathbf{J}_{i_1,j_1}$ the type $\mathbf{tp}_{\mathfrak{s}}(c, M_{i_1,j_1}, M_{i_1+1,j_1})$ is a non-forking extension of $p\}$.

3A) We say that the $\mathfrak{u}_{\mathfrak{s}}$-free triangle is strongly full <u>when</u>:

(a) as in part (3)

(b) if $i < \alpha_j(\mathbf{d})$ and $j < \beta(\mathbf{d})$ and $p \in \mathscr{S}^{\rm bs}_{\mathfrak{s}}(M^{\mathbf{d}}_{i,j})$ then for λ ordinals $i_1 \in [i, i+\lambda)$ for some $j_1 \in [j_2, \beta)$ the type $\mathbf{tp}_{\mathfrak{s}}(b^{\mathbf{d}}_{i_1,j_1}, M^{\mathbf{d}}_{i_1,j_1} M^{\mathbf{d}}_{i_1+1,j_1})$ is a non-forking extension of p and $i_1 < \alpha_{j_1}(\mathbf{d})$, of course.

3B) We say a $\mathfrak{u}_{\mathfrak{s}}$-free rectangle $\mathbf{d}$ is full [strongly full] <u>when</u> both $\mathbf{d}$ and its dual are full [strongly full] $\mathfrak{u}_{\mathfrak{s}}$-free triangles.

5.16 Observation. 1) If $\bar{M} = \langle M_\alpha : \alpha \leq \delta\rangle$ is $\leq_{\mathfrak{s}}$-increasing continuous, δ a limit ordinal and $\delta = \sup\{\alpha < \delta : M_{\alpha+1}$ or just M_β for some $\beta \in (\alpha,\delta)$, is $\leq_{\mathfrak{s}}$-universal over $M_\alpha\}$ <u>then</u> $\delta \in$ correct$(\bar{M})$.
2) If cf$(\delta_\ell) = \kappa$ and $\bar{M}^\ell = \langle M^\ell_\alpha : \alpha \leq \delta_\ell\rangle$ is $\leq_{\mathfrak{s}}$-increasing continuous and $h_\ell : \kappa \to \delta_\ell$ is increasing with $\delta_\ell = \sup(\text{Rang}(h_\ell))$ for $\ell = 0,1$ and $\varepsilon < \kappa \Rightarrow M^1_{h_1(\varepsilon)} = M^2_{h_2(\varepsilon)}$ <u>then</u> $\bar{M}^1$ is δ_1-correct iff $\bar{M}^2$ is δ_2-correct.
3) Instead of the h_1, h_2 is part (2) it suffices that $(\forall \alpha < \delta_\ell)(\exists \beta < \delta_{3-\ell})(M^\ell_\alpha \leq_{\mathfrak{s}} M^{3-\ell}_\beta)$ for $\ell = 1,2$. Also instead of "δ_ℓ-correct" we can use (Γ,δ_ℓ)-correct.
4) If $\bar{M} = \langle M_\alpha : \alpha \leq \delta\rangle$ is $\leq_{\mathfrak{s}}$-increasing continuous and $\Gamma_{\bar{M}} = \{p \in \mathscr{S}^{\rm bs}_{\mathfrak{s}}(M_\delta) : p$ does not fork over M_α for some $\alpha < \delta\}$ <u>then</u> $\bar{M}$ is δ-correct iff $\bar{M}$ is (δ,Γ)-correct.

Proof. 1) By Ax(E)(c)$^-$ and the definition of correct$(\bar{M})$.
2),3),4) Read the definitions. $\square_{5.16}$

5.17 Observation. Assume $\mathbf{d}^{\rm ver}$ is a $\mathfrak{u}_{\mathfrak{s}}$-free $(\alpha,0)$-rectangle, $\mathbf{d}_{\rm hor}$ is a $\mathfrak{u}_{\mathfrak{s}}$-free $(0,\beta)$-rectangle and $M^{\mathbf{d}^{\rm ver}}_{0,0} = M^{\mathbf{d}_{\rm hor}}_{0,0}$. <u>Then</u> there is a pair $(\mathbf{d}, f)$ such that:

(a) $\mathbf{d}$ is a strictly brimmed $\mathfrak{u}$-free (α,β)-rectangle

(b) $\mathbf{d} \restriction (0,\beta) = \mathbf{d}_{\rm hor}$

(c) f is an isomorphism from $\mathbf{d}^{\rm ver}$ onto $\mathbf{d} \restriction (\alpha, 0)$ over $M^{\mathbf{d}}_{0,0}$

(d) if α is divisible by λ and $S^{\rm ver} = \{\alpha' < \alpha : \mathbf{J}^{\mathbf{d}^{\rm ver}}_{\alpha'} = \emptyset\}$ is an unbounded subset of α of order-type divisible by λ (so in particular α is a limit ordinal) <u>then</u> we can add $\mathbf{d}$ as a triangle is full; we can add strongly full if $\alpha' < \alpha \Rightarrow \lambda = |S^{\rm ver} \cap [\alpha', \alpha' + \lambda]|$

(e) if $S_{\rm hor} := \{\beta' < \beta : \mathbf{I}^{\rm hor}_\beta = 0\}$ is an unbounded subset of β of order-type divisible by λ <u>then</u> we can add "dual($\mathbf{d}$) as a triangle is full"; and we can add strongly full if $\beta' < \beta \Rightarrow \lambda = |S_{\rm hor} \cap [\beta', \beta' + \lambda]|$.

Proof. Easy. $\square_{5.17}$

<u>5.18 Exercise</u>: Show the obvious implication concerning the notions from Definition 5.15. Let $\mathbf{d}$ be a $\mathfrak{u}$-free $(\bar{\alpha}, \beta)$-triangle and $\mathbf{e}$ be a $\mathfrak{u}$-free (α, β)-rectangle.
1) $\mathbf{d}$ strictly brimmed implies $\mathbf{d}$ is brimmed which implies $\mathbf{d}$ is weakly brimmed.
2) Like (1) replacing brimmed by universal.
3) If $\mathbf{d}$ is strictly brimmed/brimmed/weakly brimmed <u>then</u> $\mathbf{d}$ is strictly universal/universal/weakly universal.
4) If $\mathbf{d}$ is strongly full <u>then</u> $\mathbf{d}$ is full.
5) Similarly for the rectangle $\mathbf{e}$.
6) If $\mathbf{e}$ is strictly brimmed/brimmed/weakly brimmed <u>then</u> so is dual($\mathbf{e}$).
7) If $\mathbf{e}$ is strictly universal/universal/weakly universal <u>then</u> so is dual($\mathbf{e}$).

5.19 The Correctness Claim. *1) Assume $\delta < \lambda^+$ is a limit ordinal, $\bar{M}^\ell = \langle M^\ell_\alpha : \alpha \le \delta\rangle$ is $\le_{\mathfrak{s}}$-increasing continuous sequence for $\ell = 1, 2$ and $\alpha < \delta \Rightarrow M^1_\alpha \le_{\mathfrak{s}} M^2_\alpha$ and $M^1_\delta = M^2_\delta$. If $\bar{M}^1$ is δ-correct <u>then</u> $\bar{M}^2$ is δ-correct.*
2) M_δ is $(\lambda, {\rm cf}(\delta))$-brimmed over M_0; moreover over M_i for any $i < \delta$ <u>when</u>:

(a) *δ is a limit ordinal divisible by λ (the divisibility follows by clause (d))*

(b) *$\bar{M} = \langle M_\alpha : \alpha \le \delta\rangle$ is $\le_{\mathfrak{s}}$-increasing continuous*

(c) *$\bar{M}$ is (δ, Γ)-correct, so $\Gamma \subseteq \{p \in \mathscr{S}^{\rm bs}_{\mathfrak{s}}(M_\delta) : p$ does not fork over M_α for some $\alpha < \delta\}$, if Γ is equal this means $\delta \in {\rm correct}(\bar{M})$, recalling Definition 5.14(1),(6)*

(d) *if $\alpha < \delta$ and $p \in \{q \restriction M_\alpha : q \in \Gamma\} \subseteq \mathscr{S}^{\rm bs}_{\mathfrak{s}}(M_\alpha)$ <u>then</u>: for $\ge \lambda$ ordinals $\beta \in (\alpha, \delta)$ <u>there</u> is $c \in M_{\beta+1}$ such that $\mathbf{tp}_{\mathfrak{s}}(c, M_\beta, M_{\beta+1})$ is a non-forking extension of p.*

2A) M_δ is $(\lambda, {\rm cf}(\delta))$-brimmed over M_0 <u>when</u> clauses (a),(b) of part (2) holds and $\delta = \sup(S)$ where $S = \{\delta' : \delta' < \delta$ and $\bar{M} \restriction (\delta' + 1)$ satisfies clauses (a)-(d) from part (2)$\}$.

3) Assume $\mathbf{d}$ *is a* $\mathfrak{u}_{\mathfrak{s}}$*-free* $(\bar\alpha, \beta)$*-triangle,* β *is a limit ordinal,* $\bar\alpha$ *is continuous (or just* $\alpha_\beta = \alpha_\beta(\mathbf{d}) = \cup\{\alpha_{j+1} : j < \beta\}$*) and* $\bar\alpha \restriction \beta$ *is not eventually constant,*

- (a) *if* $\mathbf{d}$ *is brimmed or just weakly universal* <u>*then*</u> $\langle M^{\mathbf{d}}_{\alpha,\beta} : \alpha \le \alpha_\beta\rangle$ *is* α_β*-correct; moreover is correct for* $(\alpha_\beta, \Gamma_{\mathbf{d}})$ *recalling* $\Gamma_{\mathbf{d}} = \{p \in \mathscr{S}^{\rm bs}_{\mathfrak{s}}(M^{\mathbf{d}}_{\alpha_\beta,\beta}) : p$ *does not fork over* $M^{\mathbf{d}}_{i,j}$ *for some* $j < \beta, i < \alpha_j\}$
- (b) *if* $\mathbf{d}$ *is weakly universal and full* <u>*then*</u> $M_{\alpha_\beta,\beta}$ *is brimmed over* $M_{i,\beta}$ *for every* $i < \alpha_\beta$*.*

3A) Assume $\mathbf{d}$ *is an* $\mathfrak{u}_{\mathfrak{s}}$*-free* (α, β)*-rectangle*

- (a) *if* $\mathbf{d}$ *is brimmed or just weakly universal (see Definition 5.15(2A)) and* $\mathrm{cf}(\alpha) = \mathrm{cf}(\beta) \ge \aleph_0$ <u>*then*</u> $\langle M^{\mathbf{d}}_{i,\beta} : i \le \alpha\rangle$ *is* α*-correct and even* $(\alpha, \Gamma_{\mathbf{d}})$*-correct*
- (b) *if in clause (a),* $\mathbf{d}$ *is full* <u>*then*</u> $M^{\mathbf{d}}_{\alpha,\beta}$ *is* $(\lambda, \mathrm{cf}(\alpha))$*-brimmed over* $M_{i,\beta}$ *for* $i < \alpha$
- (c) *if* $\mathbf{d}$ *is strictly brimmed (or just strictly universal, see Definition 5.15(2)) and strongly full, (see Definition 5.15(3A),(3B)) and* $\lambda^2\omega$ *divides* α *(but no requirement on the cofinalities)* <u>*then*</u> $M^{\mathbf{d}}_{\alpha,\beta}$ *is* $(\lambda, \mathrm{cf}(\alpha))$*-brimmed over* $M^{\mathbf{d}}_{i,\beta}$ *for every* $i < \alpha$*.*

4) For $M \in K_{\mathfrak{s}}$ *there is* $N \in K_{\mathfrak{s}}$ *which is brimmed over* M *and is unique up to isomorphism over* M *(so in other words, if* M_ℓ *is* (λ, κ_ℓ)*-brimmed over* M *for* $\ell = 1, 2$ <u>*then*</u> N_1, N_2 *are isomorphic over* M*).*

Proof. 1) Assume $M^2_\delta <_{\mathfrak{s}} N$, hence $M^1_\delta <_{\mathfrak{s}} N$ so as $\delta \in \mathrm{correct}(\bar M^1)$ necessarily for some pair (p, α) we have: $\alpha < \delta$ and $p \in \mathscr{S}^{\rm bs}_{\mathfrak{s}}(M^1_\delta)$ is realized in N and does not fork over M^1_α. As $M^1_\alpha \le_{\mathfrak{s}} M^2_\alpha \le_{\mathfrak{s}} M^2_\delta = M^1_\delta$ and montonicity of non-forking it follows that "$p \in \mathscr{S}^{\rm bs}_{\mathfrak{s}}(M^2_\delta)$ does not fork over M^2_α", and, of course, p is realized in N. So (p, α) are as required in the definition of "$\delta \in \mathrm{correct}(\bar M^2)$".
2) Similar to III§4 but we give a full self-contained proof. The "moreover" can be proved by renaming.

Let $\langle \mathscr{U}_\alpha : \alpha < \delta\rangle$ be an increasing continuous sequence of subsets of λ such that $|\mathscr{U}_0| = \lambda, |\mathscr{U}_{\alpha+1} \backslash \mathscr{U}_\alpha| = \lambda$. We choose a triple $(\bar{\mathbf{a}}^\alpha, N_\alpha, f_\alpha)$ by induction on $\alpha \le \delta$ such that:

$\oplus$
- (a) $N_\alpha \in \mathfrak{K}_{\mathfrak{s}}$ and $N_0 = M_0$
- (b) f_α is a $\le_{\mathfrak{s}}$-embedding of M_α into N_α
- (c) $\langle N_\beta : \beta \le \alpha\rangle$ is $\le_{\mathfrak{s}}$-increasing continuous
- (d) $\langle f_\beta : \beta \le \alpha\rangle$ is $\subseteq$-increasing continuous and $f_0 = \mathrm{id}_{M_0}$

(e) $\bar{\mathbf{a}}^\alpha = \langle a_i : i \in \mathscr{U}_\alpha\rangle$, so $\beta < \alpha \Rightarrow \bar{\mathbf{a}}^\beta = \bar{\mathbf{a}}^\alpha \restriction \mathscr{U}_\beta$

(f) $\bar{\mathbf{a}}^\alpha$ lists the elements of N_α each appearing λ times

(g) if $\alpha = \beta + 1$ then N_α is $\leq_{\mathfrak{s}}$-universal over N_β

(h) if $\alpha = \beta + 1$ and $\mathscr{W}_\beta := \{i \in \mathscr{U}_\beta$: for some $c \in M_{\alpha+1} \backslash M_\alpha$ we have $f_\beta(\mathbf{tp}_{\mathfrak{s}}(c, M_\beta, M_\alpha)) = \mathbf{tp}_{\mathfrak{s}}(a_i, f_\beta(M_\beta), N_\beta)\}$ is not empty and $i_\beta :=$ min$(\mathscr{W}_\beta)$ then $a_{i_\beta} \in$ Rang(f_α).

There is no problem to carry the definition; and by clauses (c),(g) of $\oplus$ obviously

$\odot$ N_δ is $(\lambda, \mathrm{cf}(\delta))$-brimmed over N_0 (hence over $f_0(M_0)$).

Also by renaming without loss of generality $f_\alpha =$ id$_{M_\alpha}$ for $\alpha \leq \delta$ hence $M_\delta \leq_{\mathfrak{s}} N_\delta$.

Now if $M_\delta = N_\delta$ then by $\odot$ we are done. Otherwise by clause (c) of the assumption $\bar{M}$ is (δ, Γ)-correct hence by Definition 5.14(1), for some $c \in N_\delta \backslash f_\delta(M_\delta)$ and $\alpha_0 < \delta$ the type $\mathbf{tp}_{\mathfrak{s}}(c, M_\delta, N_\delta)$ belongs to $\Gamma \subseteq \mathscr{S}^{\mathrm{bs}}_{\mathfrak{s}}(M_\delta)$ and does not fork over M_{α_0}. As $\langle N_\beta : \beta \leq \delta\rangle$ is $\leq_{\mathfrak{s}}$-increasing continuous, for some $\alpha_1 < \delta$ we have $c \in N_{\alpha_1}$, hence for some $i_* \in \mathscr{U}_{\alpha_1}$ we have $c = a_{i_*}$. So $\alpha_2 :=$ max$\{\alpha_0, \alpha_1\} < \delta$ and by clause (d) of the assumption the set $\mathscr{W} := \{\alpha < \delta : \alpha \geq \alpha_2$ and for some $c' \in M_{\alpha+1}$ the type $\mathbf{tp}_{\mathfrak{s}}(c', M_\alpha, M_{\alpha+1})$ is a non-forking extension of $\mathbf{tp}_{\mathfrak{s}}(c, M_{\alpha_0}, N_\delta)\}$ has at least λ members and is $\subseteq [\alpha_2, \delta)$ and by the monotonicity and uniqueness properties of non-forking we have $\mathscr{W} = \{\alpha < \delta : \alpha \geq \alpha_2$ and some $c' \in M_{\alpha+1}$ realizes $\mathbf{tp}_{\mathfrak{s}}(c, M_\alpha, N_\delta)$ in $M_{\alpha+1}\}$. Now for every $\alpha \in \mathscr{W} \subseteq [\alpha_2, \delta)$ the set $\mathscr{W}_\alpha$ defined in clause (h) of $\oplus$ above is not empty, in fact, $i_* \in \mathscr{W}_\alpha$ hence $\beta \in \mathscr{W} \subseteq [\alpha_2, \delta) \Rightarrow i_\beta =$ min$(\mathscr{W}_\beta) \leq i_*$ but $|\mathscr{W}| = \lambda$, so by cardinality consideration for some $\beta_1 < \beta_2$ from $\mathscr{W}$ we have $i_{\beta_1} = i_{\beta_2}$ but $a_{i_{\beta_1}} \in$ Rang$(f_{\beta_1+1}) \subseteq$ Rang(f_{β_2}) whereas $a_{i_{\beta_2}} \notin$ Rang(f_{β_2}), contradiction.

2A) If $\alpha < \beta \in S$ then by part (2) applied to the sequence $\langle M_{\alpha+\gamma} : \gamma \leq \beta - \alpha\rangle$, the model M_β is $(\lambda, \mathrm{cf}(\beta))$-brimmed over M_α hence M_β is $\leq_{\mathfrak{s}}$-universal over M_α by 5.8(1)(c). Choose an increasing continuous sequence $\langle \alpha_\varepsilon : \varepsilon <$ cf$(\delta)\rangle$ with limit δ such that $\varepsilon <$ cf$(\delta) \Rightarrow \alpha_{\varepsilon+1} \in S$ and $\alpha_0 = 0$, so clearly $\langle M_{\alpha_\varepsilon} : \varepsilon <$ cf$(\delta)\rangle$ exemplifies that M_δ is $(\lambda,$ cf$(\delta))$-brimmed over M_0.

3) <u>Clause (a)</u>:

Note that necessarily cf$(\alpha_\beta) =$ cf(β) as $\bar{\alpha} = \langle \alpha_j : j \leq \beta\rangle$ is non-decreasing and $\bar{\alpha} \restriction \beta$ is not eventually constant.

Let $\langle \beta_\varepsilon : \varepsilon <$ cf$(\alpha_\beta)\rangle, \langle \gamma_\varepsilon : \varepsilon <$ cf$(\alpha_\beta)\rangle$ be increasing continuous sequences of ordinals with limit β, α_β respectively such that $\gamma_\varepsilon \leq \alpha_{\beta_\varepsilon}$ for every $\varepsilon <$ cf(α_β).

We now choose a pair $(i_\varepsilon, j_\varepsilon)$ by induction on $\varepsilon <$ cf(α_β) such that:

$\odot$ (a) $j_\varepsilon < \beta$ is increasing continuous with ε

(b) $i_\varepsilon \leq \alpha_{j_\varepsilon}$ is increasing continuous with ε

(c) if $\varepsilon = \zeta + 1$ then $M^{\mathbf{d}}_{i_\varepsilon, j_\varepsilon}$ is $\leq_{\mathfrak{s}}$-universal over $M^{\mathbf{d}}_{i_\zeta, j_\zeta}$

(d) if $\varepsilon = \zeta + 1$ then $i_\varepsilon > \gamma_\varepsilon, j_\varepsilon > \beta_\varepsilon$.

There is no problem to carry the definition as $\mathbf{d}$ is weakly universal, see Definition 5.15(2A) and $\bar{\alpha}$ not eventually constant. Now the sequence $\langle i_\varepsilon : \varepsilon < \mathrm{cf}(\alpha_\beta)\rangle$ is increasing with limit α_β (by clause $\odot$(d)), and $\langle j_\varepsilon : \varepsilon < \beta\rangle$ is an increasing continuous sequence and has limit β (as $\langle \alpha_j : j < \beta\rangle$ is not eventually constant), hence

(∗) $\langle M^{\mathbf{d}}_{i_\varepsilon, j_\varepsilon} : \varepsilon < \mathrm{cf}(\beta)\rangle$ is $\le_{\mathfrak{s}}$-increasing continuous with union $M^{\mathbf{d}}_{\alpha_\beta,\beta}$.

Let $(i_{\mathrm{cf}(\beta)}, j_{\mathrm{cf}(\beta)}) := (\alpha_\beta, \beta)$.

So by $\odot(c) + (*)$ it follows that $\langle M^{\mathbf{d}}_{i_\varepsilon, j_\varepsilon} : \varepsilon \le \mathrm{cf}(\beta)\rangle$ satisfies the assumptions of claim 5.16(1), hence its conclusion, i.e. the sequence $(\langle M^{\mathbf{d}}_{i_\varepsilon, j_\varepsilon} : \varepsilon \le \mathrm{cf}(\beta)\rangle$ is $\mathrm{cf}(\beta)$-correct.

We shall apply part (1) of the present claim 5.19. Now the pair $(\langle M^{\mathbf{d}}_{i_\varepsilon, j_\varepsilon} : \varepsilon \le \mathrm{cf}(\beta)\rangle, \langle M^{\mathbf{d}}_{i_\varepsilon,\beta} : \varepsilon \le \mathrm{cf}(\beta)\rangle)$ satisfies its assumptions hence its conclusion holds and it says that $\langle M^{\mathbf{d}}_{i_\varepsilon,\beta} : \varepsilon \le \mathrm{cf}(\beta)\rangle$ is $\mathrm{cf}(\beta)$-correct. As $\langle i_\varepsilon : \varepsilon \le \mathrm{cf}(\beta)\rangle$ is increasing continuous with last element $i_\varepsilon = \alpha_\beta$ and also $\langle M^{\mathbf{d}}_{i,\beta} : i \le \alpha_\beta\rangle$ is $\le_{\mathfrak{s}}$-increasing continuous also $\langle M^{\mathbf{d}}_{i,\beta} : i \le \alpha_\beta\rangle$ is α_β-correct, by Observation 5.16(2), as required.

Clause (b):

We shall apply part (2) of the present claim on the sequence $\langle M^{\mathbf{d}}_{i,\beta} : i \le \alpha_\beta\rangle$ and $\Gamma = \Gamma_{\mathbf{d}} := \{p \in \mathscr{S}^{\mathrm{bs}}_{\mathfrak{s}}(M^{\mathbf{d}}_{\alpha_\beta,\beta}) : p$ does not fork over $M^{\mathbf{d}}_{i,j}$ for some $i < \alpha_j, j < \beta\}$. By the definition of an $\mathfrak{s}$-free triangle it is $\le_{\mathfrak{s}}$-increasing continuous hence clause (b) of part (2) is satisfied. As $\mathbf{d}$ is full by clause (a) of Definition 5.15(3) the ordinal $\alpha_\beta = \alpha_\beta(d)$ is divisible by λ, i.e. clause (a) of part (2) holds. Clause (c) of the assumption of part (2) is satisfied because we have proved clause (a) here.

As for clause (d) of part (2) let $i_1 < \alpha_\beta$ and $p_1 \in \mathscr{S}^{\mathrm{bs}}_{\mathfrak{s}}(M^{\mathbf{d}}_{i_1,\beta})$ be given; let $p_2 \in \Gamma_{\mathbf{d}} = \mathscr{S}^{\mathrm{bs}}_{\mathfrak{s}}(M^{\mathbf{d}}_{\alpha_\beta,\beta})$ be a non-forking extension of p_1. By the definition of $\Gamma_{\mathbf{d}}$, see part (2) we can find $j_2 < \beta$ and $i_2 \le \alpha_{j_2}$ such that p_2 does not fork over $M^{\mathbf{d}}_{i_2,j_2}$. By monotonicity, without loss of generality $i_2 \ge i_1$ and $i_2 < \alpha_{j_2}$. As $\mathbf{d}$ is full (see clause (b) of Definition 5.15(3)) we can find $S \subseteq [i_2, \alpha_\beta)$ of order type $\ge \lambda_{\mathfrak{s}}$ such that for each $i \in S$ an ordinal $j_*(i) < \beta$ satisfying $j_*(i) > j_2$ and an element $c \in \mathbf{J}^{\mathbf{d}}_{i,j_*(i)}$ such that $i < \alpha_{j_*(i)}$ and $\mathbf{tp}_{\mathfrak{s}}(c, M^{\mathbf{d}}_{i,j_*(i)}, M^{\mathbf{d}}_{i+1,j_*(i)})$ is a non-forking extension of $p_2 \restriction M^{\mathbf{d}}_{i_2,j_2}$. So by the definition of $\mathfrak{u}_{\mathfrak{s}}$ we have $\mathbf{J}^{\mathbf{d}}_{i,j_*(i)} = \{c\}$ and so by the definition of "$\mathbf{d}$ is a $\mathfrak{u}_{\mathfrak{s}}$-free triangle" we have $(M^{\mathbf{d}}_{i,j_*(i)}, M^{\mathbf{d}}_{i+1,j_*(i)}, c) \le^1_{\mathfrak{u}} (M^{\mathbf{d}}_{i,\beta}, M^{\mathbf{d}}_{i+1,\beta}, c)$. Hence $(M^{\mathbf{d}}_{i,\beta}, M^{\mathbf{d}}_{i+1,\beta}, c)$ realizes a non-forking extension of $p_2 \restriction M_{i_2,j_2}$, hence, by the uniqueness of non-forking extensions the element c realizes $p_2 \restriction M^{\mathbf{d}}_{i,\beta}$. So S is as required in clause (d) of the assumption of part (2).

So all the assumptions of part (2) applied to the sequence $\langle M^{\mathbf{d}}_{\alpha,\beta} : \alpha \le \alpha_\beta\rangle$ and the set $\Gamma_{\mathbf{d}}$ are satisfied hence its conclusion which says that $M^{\mathbf{d}}_{\alpha_\beta,\beta}$ is $(\lambda, \mathrm{cf}(\alpha_\beta))$-brimmed over $M_{i,\beta}$ for every $i < \alpha_\beta$. So we are done proving clause (b) hence part (3).

3A) We prove each clause.

Clause (a):

So $\theta :=$ cf$(\alpha) =$ cf(β). Let $\langle \alpha_\varepsilon : \varepsilon < \theta\rangle$ be an increasing sequence of ordinals with limit α and $\langle \beta_\varepsilon : \varepsilon < \theta\rangle$ be an increasing sequence of ordinals with limit β. Now for each $\varepsilon < \theta$ we can find $i \in (\alpha_\varepsilon, \alpha)$ and $j \in (\beta_\varepsilon, \beta)$ such that $M^{\mathbf{d}}_{i,j}$ is $\le_{\mathfrak{s}}$-universal over $M^{\mathbf{d}}_{\alpha_\varepsilon,\beta_\varepsilon}$; this holds as we are assuming $\mathbf{d}$ is weakly universal, see Definition 5.15(2A).

By monotonicity without loss of generality $i \in \{\alpha_\zeta : \zeta \in (\varepsilon, \theta)\}$ and $j \in \{\beta_\zeta : \zeta \in (\varepsilon, \theta)\}$. So without loss of generality $M^{\mathbf{d}}_{\alpha_{\varepsilon+1},\beta_{\varepsilon+1}}$ is $\le_{\mathfrak{s}}$-universal over $M^{\mathbf{d}}_{\alpha_\varepsilon,\beta_\varepsilon}$ for $\varepsilon < \theta$. Let $\alpha_\theta := \alpha, \beta_\theta = \beta$.

Hence by Observation 5.16(1) we have $\theta \in$ correct$(\langle M^{\mathbf{d}}_{\alpha_\varepsilon,\beta_\varepsilon} : \varepsilon \le \theta\rangle)$ which means $\theta \in$ correct$_{\Gamma_{\mathbf{d}}}(\langle M^{\mathbf{d}}_{\alpha_\varepsilon,\beta_\varepsilon} : \varepsilon \le \theta\rangle)$, see 5.16(4). So by part (1) also $\theta \in$ correct$_{\Gamma_{\mathbf{d}}}(\langle M^{\mathbf{d}}_{\alpha_\varepsilon,\beta} : \varepsilon \le \theta\rangle)$, hence by Observation 5.16(2) also $\alpha \in$ correct$_{\Gamma_{\mathbf{d}}}(\langle M^{\mathbf{d}}_{i,\beta} : i \le \alpha\rangle)$, as required.

Clause (b):

We can apply part (2) of the present claim to the sequence $\langle M^{\mathbf{d}}_{i,\beta} : i \le \alpha\rangle$. This is similar to the proof of clause (b) of part (3), alternatively letting $\alpha'_j = \sup\{\alpha_\varepsilon : \varepsilon < \theta$ and $\beta_\varepsilon \le j\}$ for $j < \beta$ and $\bar\alpha' = \langle \alpha'_j : j \le \beta\rangle$ use part (3) for the $\mathfrak{u}$-free triangle $\mathbf{d} \restriction (\bar\alpha, \beta)$, i.e. $\bar M^{\mathbf{d}} = \langle M_{i,j} : j \le \beta$ and $i \le \alpha'_j\rangle$, etc.; this applies to clause (a), too.

Clause (c):

Note that "cf$(\alpha) =$ cf(β)" is not assumed.

We use part (2A) of the present claim, but we elaborate. Let $S := \{\alpha' < \alpha : \alpha'$ is divisible by λ and has cofinality cf$(\beta)\}$.

Now S is a subset of α, unbounded (as for every $i < \alpha$ we have $i + \lambda(\mathrm{cf}(\beta)) \in S \cup \{\alpha\}$ and $i + \lambda(\mathrm{cf}(\beta)) \le i + \lambda^2 < i + \lambda^2\omega \le \alpha$), hence it is enough to show that $i_1 < i_2 \in S \Rightarrow M^{\mathbf{d}}_{i_2,\beta}$ is brimmed over $M^{\mathbf{d}}_{i_1,\beta}$.

Now this follows by clause (b) of part (3A) which we have just proved applied to $\mathbf{d}' = \mathbf{d} \restriction (i_2, \beta)$, it is a $\mathfrak{u}$-free (i_2, β)-rectangle, it is strongly full hence full and cf$(i_2) =$ cf$(\beta) \ge \aleph_0$. So the assumptions of part (2A) holds hence its conclusion so we are done.

4) Let κ_1, κ_2 be regular $\le \lambda$ and choose $\alpha_\ell = \lambda^2 \times \kappa_\ell$, for $\ell = 1, 2$. Let $M \in K_{\mathfrak{s}}$ and define a $\mathfrak{u}$-free $(\alpha_1, 0)$-rectangle by $M^{\mathbf{d}^{\mathrm{ver}}_0}_{(i,0)} = M$ for $i \le \alpha_1$ and $\mathbf{J}^{\mathbf{d}^{\mathrm{ver}}_0}_{(i,0)} = \emptyset$ for $i < \alpha_1$.

Define a $\mathfrak{u}$-free $(0,\alpha_2)$-rectangle $\mathbf{d}_{\rm hor}$ by $M^{\mathbf{d}_{\rm hor}}_{(0,j)} = M$ for $j \le \alpha_2$ and $\mathbf{I}^{\mathbf{d}}_{(0,j)} = \emptyset$ for $j < \alpha_2$. By Observation 5.17 there is a strongly full strictly brimmed $\mathfrak{u}$-free (α_1,α_2)-rectangle $\mathbf{d}$ such that its dual is strongly full too (and automatically strictly brimmed recalling 5.18(6)).

We can apply clause (c) of part (3A) with $(\mathbf{d},\alpha_1,\alpha_2)$ here standing for $(\mathbf{d},\alpha,\beta)$ there; so we can conclude in particular that M_{α_1,α_2} is $(\lambda,{\rm cf}(\alpha_1))$-brimmed over $M^{\mathbf{d}}_{0,\alpha_2}$ hence over $M_{0,0} = M$. But $\mathfrak{u}_{\mathfrak{s}}$ is self-dual so dual$(\mathbf{d})$ is a $\mathfrak{u}_{\mathfrak{s}}$-free (α_2,α_1)-rectangle, and by the choice of $\mathbf{d}$ (recalling 5.17(d)) it is still strongly full and, e.g. by 5.18(7) is universal. So applying clause (c) of part (3A) we get that $M^{{\rm dual}(\mathbf{d})}_{\alpha_2,\alpha_1}$ is $(\lambda,{\rm cf}(\alpha_2))$-brimmed over $M^{{\rm dual}(\mathbf{d})}_{0,\alpha_1}$ hence over $M^{{\rm dual}(\mathbf{d})}_{0,0} = M^{\mathbf{d}}_{0,0} = M$. However $M^{{\rm dual}(\mathbf{d})}_{\alpha_2,\alpha_1} = M^{\mathbf{d}}_{\alpha_1,\alpha_2}$ so this model $\le_{\mathfrak{s}}$-extend M and is $(\lambda,{\rm cf}(\alpha_\ell))$-brimmed over it for $\ell = 1,2$; this means (λ,κ_ℓ)-brimmed over M. So as for each regular $\kappa \le \lambda$, the "(λ,κ)-brimmed model over M for some regular $\kappa \le \lambda$" is unique up to isomorphism over M we conclude that the brimmed model over M is unique, so we are done. $\square_{5.19}$

5.20 Claim. *If $M_1 \le_{\mathfrak{s}} M_2$ are brimmed and $p_i \in \mathscr{S}^{\rm bs}_{\mathfrak{s}}(M_2)$ does not fork over M_1 for $i < i_* < \lambda_{\mathfrak{s}}$ <u>then</u> for some isomorphism π from M_2 onto M_1 we have $i < i_* \Rightarrow \pi(p_i) = p_i \restriction M_1$.*

Proof. Easy, by 5.19(4) and 5.8(3), i.e. as in IV.705-stg.9. $\square_{5.20}$

* * *

Another way to deal with disjointness is through reduced triples (earlier we have used 1.10, 1.11, 4.25 (with some repetitions).

5.21 Definition. 1) $K^{3,{\rm rd}}_{\mathfrak{s}}$ is the class of triples $(M,N,a) \in K^{3,{\rm bs}}_{\mathfrak{s}}$ which are reduced[24] which means: if $(M,N,a) \le_{\rm bs} (M_1,N_1,a) \in K^{3,{\rm bs}}_{\mathfrak{s}}$ then $N \cap M_1 = M$.
2) We say that $\mathfrak{s}$ has existence for $K^{3,{\rm rd}}_{\mathfrak{s}}$ <u>when</u> for every $M \in K_{\mathfrak{s}}$ and $p \in \mathscr{S}^{\rm bs}_{\mathfrak{s}}(M)$ there is a pair (N,a) such that the triple $(M,N,a) \in K^{3,{\rm rd}}_{\mathfrak{s}}$ realizes p, i.e. $p = \mathbf{tp}_{\mathfrak{s}}(a,M,N)$.
3) Let $\xi^{\rm rd}_{\mathfrak{s}}$ be the minimal ξ from Claim 5.22(4) below for $M \in K^{3,{\rm bs}}_{\mathfrak{s}}$ which is superlimit.
3A) For $M \in K_{\mathfrak{s}}$, let $\xi^{\rm rd}_{\mathfrak{s},M} = \xi^{\rm rd}_M$ be the minimal $\xi < \lambda^+_{\mathfrak{s}}$ in 5.22(4) below for M <u>when</u> it exists, ∞ otherwise (well defined, i.e. $< \infty$ if $\mathfrak{s}$ has existence for $K^{3,{\rm rd}}_{\mathfrak{s}}$).

[24] This is different from our choice in Definition VI.E46-1a.34(2), <u>but</u> here this is for a given almost good λ-frame, there it is for a λ-a.e.c. $\mathfrak{K}$.

5.22 Claim. *1) For every $(M,N,a) \in K^{3,\mathrm{bs}}_{\mathfrak{s}}$ there is $(M_1,N_1,a) \in K^{3,\mathrm{rd}}_{\mathfrak{s}}$ such that $(M,N,a) \le_{\mathrm{bs}} (M_1,N_1,a)$ and moreover M_1,N_1 are brimmed over M,N respectively.*
2) $K^{3,\mathrm{rd}}_{\mathfrak{s}}$ is closed under increasing unions of length $< \lambda^+$, i.e. if $\delta < \lambda^+$ is a limit ordinal and $(M_\alpha,N_\alpha,a) \in K^{3,\mathrm{rd}}_{\mathfrak{s}}$ is $\le^{\mathrm{bs}}_{\mathfrak{s}}$-increasing with $\alpha(< \delta)$ and $M_\delta := \cup\{M_\alpha : \alpha < \delta\}$ and $N_\delta := \cup\{N_\alpha : \alpha < \delta\}$ then $(M_\delta,N_\delta,a) \in K^{3,\mathrm{rd}}_{\mathfrak{s}}$ and $\alpha < \delta \Rightarrow (M_\alpha,N_\alpha,a) \le^{\mathrm{bs}}_{\mathfrak{s}} (M_\delta,N_\delta,a)$.
3) If $\mathfrak{K}_{\mathfrak{s}}$ is categorical (in λ) then $\mathfrak{s}$ has existence for $K^{3,\mathrm{rd}}_{\mathfrak{s}}$.
4) For every $M \in K_{\mathfrak{s}}$ and $p \in \mathscr{S}^{\mathrm{bs}}_{\mathfrak{s}}(M)$ there are $\xi < \lambda^+$, a $\le_{\mathfrak{s}}$-increasing continuous $\bar M = \langle M_\alpha : \alpha \le \xi\rangle$ and $\bar a = \langle a_\alpha : \alpha < \xi\rangle$ such that $M_0 = M, M_\xi$ is brimmed over M_0 and each $(M_\alpha,M_{\alpha+1},a_\alpha)$ is a reduced member of $K^{3,\mathrm{bs}}_{\mathfrak{s}}$ and a_0 realizes p in M_1, provided that[25] $\mathfrak{s}$ is categorical or M is brimmed (equivalently superlimit) or $\mathfrak{s}$ has existence for $K^{3,\mathrm{rd}}_{\mathfrak{s}}$.
5) If $(M_1,N_1,a) \in K^{3,\mathrm{rd}}_{\mathfrak{s}}$ and $(M_1,N_1,a) \le_{\mathrm{bs}} (M_2,N_2,a)$ then $M_2 \cap N_1 = M_1$.

Proof. 1),2),3) Easy, or see details in [Sh:F841].
4) As in the proof of 7.10(2B) using Fodor lemma; for the "M brimmed" case, use the moreover from part (1).
5) By the definitions. $\square_{5.22}$

5.23 Conclusion. (Disjoint amalgamation) Assume $\mathfrak{K}_{\mathfrak{s}}$ is categorical or just has existence for $K^{3,\mathrm{rd}}_{\mathfrak{s}}$ recalling 5.22(3). If $(M,N_\ell,a_\ell) \in K^{3,\mathrm{bs}}_{\mathfrak{s}}$ for $\ell = 1,2$ and $N_1 \cap N_2 = M$ then there is $N_3 \in K_{\mathfrak{s}}$ such that $(M,N_\ell,a_3) \le_{\mathrm{bs}} (N_{3-\ell},N_3,a_\ell)$ for $\ell = 1,2$.

Hence $\mathfrak{s}$ has disjointness, see definition 5.5.

Proof. Straight by 5.22(4) similarly to Observation 5.17 using 5.22(5), of course. $\square_{5.23}$

5.24 Question: Is 5.23 true without categoricity (and without assuming existence for $K^{3,\mathrm{rd}}_{\mathfrak{s}}$)?

5.25 Remark. So we can redefine $\mathfrak{u}$ such that the amalgamation is disjoint by restricting ourselves to $\mathfrak{s}_{[M]}, M \in K_{\mathfrak{s}}$ superlimit or assuming $\mathfrak{s}$ has existence for $K^{3,\mathrm{rd}}_{\mathfrak{s}}$.

[25] Why not $\xi \le \lambda$? The bookkeeping is O.K. but then we have to use Ax(E)(c) in the end, but see Exercise 5.26.

We can work with $\mathfrak{u}$ which includes disjointness so this enters the definition of $K^{3,\mathrm{up}}_{\mathfrak{s}}$ defined in 6.4, so this is a somewhat different property or, as we prefer, we ignore this using $=_\tau$ as in §1.

5.26 Exercise: If M is superlimit and $(M,N,a) \in K^{3,\mathrm{bs}}_{\mathfrak{s}}$ then for some $\le_{\mathfrak{s}}$-increasing continuous sequence $\bar M = \langle M_\alpha : \alpha < \lambda\rangle$ and $\bar a = \langle a_\alpha : \alpha < \lambda\rangle, \mathbf{d}$ we have $M = M_0 \le_{\mathfrak{s}} N \le_{\mathfrak{s}} M_\lambda$ and each $(M_\alpha, M_{\alpha+1}, a_\alpha) \in K^{3,\mathrm{bs}}_{\mathfrak{s}}$ is reduced and $a = a_0$ and M_λ is brimmed over M_0.

[Hint: Let $\bar{\mathscr{U}} = \langle \mathscr{U}_\alpha : \alpha \le \lambda\rangle$ be an increasing continuous sequence of subsets of λ such that $|\mathscr{U}_0| = \lambda = |\mathscr{U}_{\alpha+1} \backslash \mathscr{U}_0|$ and $\min(\mathscr{U}_\alpha) \ge \alpha$ for $\alpha < \lambda$ and $\mathscr{U}_\lambda = \lambda$.

Let $\alpha_i = i$ for $i \le \lambda$ and $\bar\alpha^i = \langle \alpha_j : j \le i\rangle$. We choose pairs $(\mathbf{d}_\beta, \bar p^\beta)$ by induction on $\beta \le \lambda$ such that

$\circledast$ (a) $\mathbf{d}_\beta$ is a $\mathfrak{u}$-free $(\bar\alpha^\beta, \beta)$-triangle

(b) $M^{\mathbf{d}_\beta}_{i+1,j+1}$ is $\le_{\mathfrak{s}}$-universal over $M^{\mathbf{d}_\beta}_{i+1,j} \cup M^{\mathbf{d}_\beta}_{i,j+1}$ when $j < \beta$ and $i < \alpha_j$ and $M^{\mathbf{d}_\beta}_{0,j+1}$ is $\le_{\mathfrak{s}}$-universal over $M^{\mathbf{d}_\beta}_{0,j}$ when $j < \beta$

(c) $\bar p^\beta = \langle p_i : i \in \mathscr{U}_\alpha\rangle$ list $\cup\{\mathscr{S}^{\mathrm{bs}}_{\mathfrak{s}}(M^{\mathbf{d}_\beta}_{i,j}) : j \le \beta, i \le \alpha_j\}$ each appearing λ times

(d) if $\beta = 2\alpha+1, \alpha \in \mathscr{U}_\varepsilon$ then $\mathbf{J}^{\mathbf{d}_\beta}_{2\alpha,\beta} \ne \emptyset$ and letting a be the unique member of $\mathbf{J}^{\mathbf{d}_\beta}_{2\alpha,\beta}$, the type $\mathbf{tp}_{\mathfrak{s}}(a, M^{\mathbf{d}_\beta}_{2\alpha,\beta}, M^{\mathbf{d}_\beta}_{p,\beta})$ is a non-forking extension of p_α

(e) if $\beta = 2\alpha+2$ and $\alpha \in \mathscr{U}_\varepsilon$ then there is $(M'_\beta, N'_\beta, \mathbf{J}_\beta) \in K^{3,\mathrm{rd}}_{\mathfrak{s}}$ such that $(M^{\mathbf{d}_\beta}_{\varepsilon,2\alpha+1}, M^{\mathbf{d}_\beta}_{\varepsilon+1,2\alpha+1}, \mathbf{J}^{\mathbf{d}_\beta}_{\varepsilon,2\alpha+1}) \le^{\mathrm{bs}}_{\mathfrak{s}} (M'_\beta, N'_\beta, \mathbf{J}_\beta) \le^{\mathrm{bs}}_{\mathfrak{s}} (M^{\mathbf{d}_\beta}_{\varepsilon,\beta}, M^{\mathbf{d}_\beta}_{\varepsilon+1,\beta}, \mathbf{J}^{\mathbf{d}_\beta}_{\varepsilon,\beta})$]

(f) $M^{\mathbf{d}_0}_{0,0} = M$ and $(M^{\mathbf{d}_1}_{0,1}, M^{\mathbf{d}_1}_{1,1}, \mathbf{J}^{\mathbf{d}_1}_{0,1}) = (M, N, \{a\})$.

Now $\langle M^{\mathbf{d}_\lambda}_{i,\lambda} : i \le \lambda\rangle$ is as required except "$M = M_{0,\lambda} \le_{\mathfrak{s}} N \le_{\mathfrak{s}} M_{\lambda,\lambda}$". But $(M,N,\{a\}) = (M^{\mathbf{d}_1}_{0,1}, M^{\mathbf{d}_1}_{1,1}, \{a\}) \le^1_{\mathfrak{u}} (M^{\mathbf{d}_\lambda}_{0,\lambda}, M^{\mathbf{d}_\lambda}_{1,\lambda}, \mathbf{J}^{\mathbf{d}_\lambda}_{0,\lambda})$, that is $(M,N,a) \le_{\mathrm{bs}} (M^{\mathbf{d}_\lambda}_{0,\lambda}, a)$ and both M and $M^{\mathbf{d}_\lambda}_{0,\lambda}$ are brimmed equivalently superlimit, hence by 5.8(3) there is an isomorphism π from M onto M onto $M^{\mathbf{d}_\lambda}_{0,\lambda}$ mapping $\mathbf{tp}(a,M,N)$ to $\mathbf{tp}(a, M^{\mathbf{d}_\lambda}_{0,\lambda}, M^{\mathbf{d}_\lambda}_{1,\lambda})$. Recalling $M^{\mathbf{d}_\lambda}_{\lambda,\lambda}$is brimmed over $M^{\mathbf{d}_\lambda}_{0,\lambda}$ we can extend π to a $\le_{\mathfrak{s}}$-embedding π^+ of N into $M^{\mathbf{d}_\lambda}_{\lambda,\lambda}$ mapping a to itself, so renaming we are done.]

§6 Density of weak version of uniqueness

We would like to return to "density of $K^{3,\mathrm{uq}}_{\mathfrak{s}}$", where $\mathfrak{s}$ is a good λ-frame or just almost good λ-frames, i.e. to eliminate the (weak) extra set theoretic assumption in the non-structure results from failure of density of $K^{3,\mathrm{uq}}_{\mathfrak{s}}$. But we start with a notion $K^{3,\mathrm{up}}_{\mathfrak{s}}$, weaker then $K^{3,\mathrm{uq}}_{\mathfrak{s}}$ related to weak non-forking relations defined later in Definition 7.18. Defining weak non-forking we shall waive uniqueness, but still can "lift $\mathfrak{u}_{\mathfrak{s}}$-free $(\alpha,0)$-rectangles". We now look for a dichotomy - either a non-structure results applying the theorems of §2 or actually §3, or density of $K^{3,\mathrm{up}}_{\mathfrak{s}}$. Using the last possibility in the subsequent sections §7, §8 we get a similar dichotomy with $K^{3,\mathrm{uq}}_{\mathfrak{s}}$.

It turns out here that what we prove is somewhat weaker than density for $K^{3,\mathrm{up}}_{\mathfrak{s}}$ in some ways. Mainly we prove it for the $K^{3,\mathrm{up}}_{\mathfrak{s},\xi}$ version for each $\xi \le \lambda^+$. Actually for every $M \in K_{\mathfrak{s}}$ and $p \in \mathscr{S}^{\mathrm{bs}}_{\mathfrak{s}}(M)$ what we find is a triple $(M_1, N_1, a) \in K^{3,\mathrm{up}}_{\mathfrak{s}}$ such that $M \le_{\mathfrak{s}} M_2$ and the type $\mathbf{tp}_{\mathfrak{s}}(a, M_1, N_1)$ is a non-forking extension of p; not a serious difference when $\mathfrak{s}$ is categorical which is reasonable for our purposes. Eventually in this section we have to use $\mathfrak{K}_{\mathfrak{s}}$ with fake equality (to apply 3.16), this is justified in 6.18.

Discussion: Why do we deal with $K^{3,\mathrm{up}}_{\mathfrak{s},\xi}$ for $\xi \le \lambda^+$ rather than with $K^{3,\mathrm{up}}_{\mathfrak{s}}$? The point is that, e.g. in the weak/semi/vertical uq-invariant coding property in §3 (see Definitions 3.2, 3.14, 3.10), given $(\bar{M}, \bar{\mathbf{J}}, \mathbf{f}) \in K^{\mathrm{qt}}_{\mathfrak{u}}$ and $(M_{\alpha(0)}, N_0, \mathbf{I}) \in \mathrm{FR}^1_{\mathfrak{u}}$, for a club of $\delta < \partial$ we promise the existence of a $\mathfrak{u}$-free $(\alpha,0)$-rectangle $\mathbf{d}_\delta$, which is O.K. for every N_δ such that $(M_{\alpha(*)}, N_0, \mathbf{I}) \le^1_{\mathfrak{u}} (M_\delta, N_\delta, \mathbf{I})$. So the failure gives (not much more than) that for every $\mathfrak{u}$-free $(\alpha,0)$-rectangle $\mathbf{d}$ and $p \in \mathscr{S}^{\mathrm{bs}}_{\mathfrak{s}}(M^{\mathbf{d}}_{\alpha,0})$ there is a pair $(N, \mathbf{I})$ such that $(M, N, \mathbf{I})$ realizes p and $\mathbf{d}$ is what we call uq-orthogonal to $(M, N, \mathbf{I})$. We like to invert the quantifiers, i.e. "for $p \in \mathscr{S}^{\mathrm{bs}}_{\mathfrak{s}}(M)$ there is $(N, \mathbf{I})$ such that for every $\mathbf{d}$....". Of course, we assume categoricity (of $K_{\mathfrak{s}}$, that is in λ), but we need to use a "universal $\mathbf{d}$". This is guaranteed by 6.12 for $K^{3,\mathrm{up}}_{\mathfrak{s},\xi}$ for any $\xi \le \lambda^+$ (but we have to work more for $\xi = \lambda^+$, i.e. for all $\xi < \lambda^+$ at once, i.e. for $K^{3,\mathrm{up}}_{\mathfrak{s}}$).

6.1 Hypothesis. 1) As in 5.1 and for transparency $\mathfrak{s}$ has disjointness.
2) $\mathfrak{u} = \mathfrak{u}_{\mathfrak{s}}$, see Definition 5.10, Claim 5.11, so $\partial = \lambda^+$.

6.2 Claim. *1) For almost$_2$ all $(M, \bar{\mathbf{J}}, \mathbf{f}) \in K^{\mathrm{qt}}_{\mathfrak{u}}$ the model M_∂ belongs to $K^{\mathfrak{s}}_{\lambda^+}$ and is saturated (above λ).*
2) If $\mathfrak{s}$ has the fake equality $=_\tau$ (e.g. $\mathfrak{s} = \mathfrak{t}'$ where $\mathfrak{t}$ is an almost good λ-frame and $\mathfrak{t}'$ is defined as in 4.25(1), $\mathfrak{u} = \mathfrak{u}^1_{\mathfrak{s}}$, see 5.10,5.11, then for some $\mathfrak{u} - 0$-appropriate

$\mathfrak{h}$, *if* $\langle(\bar{M}^\alpha, \bar{\mathbf{J}}^\alpha, \mathbf{f}^\alpha) : \alpha < \partial^+_{\mathfrak{u}}\rangle$ *is* $\le^{\rm qt}_u$*-increasing continuous and obeys* $\mathfrak{h}$, *then* $M = \cup\{M^\alpha_j : \alpha < \partial^+_{\mathfrak{u}}\rangle$ *is* $=_\tau$*-fuller.*

Remark. 1) See Definition 1.22.
2) Compare with 4.30.

Proof. 1) We choose $\mathfrak{g}$ as in Definition 1.22(2) such that:

- $(*)_1$ if $S \subseteq \partial$ is a stationary subset of ∂ and the pair $((\bar{M}^1, \bar{\mathbf{J}}^1, \mathbf{f}^1), (\bar{M}^2, \bar{\mathbf{J}}^2, \mathbf{f}^2))$ strictly S-obeys $\mathfrak{g}$ then:
  - $\odot$ for some club E of ∂ for every $\delta \in S \cap E$, we have
    - (a) if $i < \mathbf{f}^1(\delta)$ then $M^2_{\delta+i+1}$ is $\le_{\mathfrak{s}}$-universal over $M^1_{\delta+i+1} \cup M^2_{\delta+i}$
    - (b) $\mathbf{f}^2(\delta)$ is $> \mathbf{f}^1(\delta)$ and is divisible by λ^2
    - (c) if $i \in [\mathbf{f}^1(\delta), \mathbf{f}^2(\delta))$ then
      - (α) $M^2_{\delta+i+1}$ is $\le_{\mathfrak{s}}$-universal over $M^2_{\delta+i}$
      - (β) if $p \in \mathscr{S}^{\rm bs}_{\mathfrak{s}}(M^2_{\delta+i})$ then for λ ordinals $i_1 \in [i, i+\lambda)$ the type $\mathbf{tp}_{\mathfrak{s}}(a_{\mathbf{J}^2_{\delta+i}}, M^2_{\delta+i}, M^2_{\delta+i+1})$ is a non-forking extension of p where $b_{\mathbf{J}^2_{\delta+i}}$ is the unique member of $\mathbf{J}^2_{\delta+i}$.

We can find such $\mathfrak{g}$. Now

- $(*)_2$ assume $\langle(\bar{M}^\alpha, \bar{\mathbf{J}}^\alpha, \mathbf{f}^\alpha) : \alpha \le \delta\rangle$ is $\le^{\rm qt}_{\mathfrak{u}}$-increasing continuous (see Definition 1.15(4A)) and obey $\mathfrak{g}$ (i.e. for some[26] stationary $S \subseteq \partial$ for unboundedly many $\alpha < \delta$). Then M^δ_∂ is saturated above λ.

[Why? Let $\kappa = \text{cf}(\delta)$, of course $\aleph_0 \le \kappa \le \lambda^+$.

We can find an increasing continuous sequence $\langle \alpha_\varepsilon : \varepsilon < \text{cf}(\delta) = \kappa\rangle$ of ordinals with limit δ such that:

- $(*)_3$ if $\varepsilon = \zeta + 1 < \kappa$ and ε is an even ordinal then $\alpha_{\varepsilon+1} = \alpha_\varepsilon + 1$ and letting $\alpha_{\text{cf}(\delta)} = \delta$ the pair $\langle(\bar{M}^{\alpha_\varepsilon}, \bar{\mathbf{J}}^{\alpha_\varepsilon}, \mathbf{f}^{\alpha_\varepsilon}), (\bar{M}^{\alpha_{\varepsilon+1}}, \bar{\mathbf{J}}^{\alpha_{\varepsilon+1}}, \mathbf{f}^{\alpha_{\varepsilon+1}})\rangle$ does S-obey $\mathfrak{g}$.

[26] we can use a decreasing sequence of S's but then we really use the last one only, the point is that $\mathfrak{g}$ treat each $\delta \in S$ in the same way (rather than dividing it according to tasks, a reasonable approach, but not needed here

Now clearly

$(*)_4$ for every $\varepsilon < \lambda^+$ satisfying $\varepsilon \le \kappa$ for some club $E = E_\varepsilon$ of λ^+, for every $\delta \in S \cap E$ we have:

- (a) $\langle \mathbf{f}^{\alpha_\zeta}(\delta) : \zeta \le \varepsilon \rangle$ is non-decreasing and is continuous
- (b) $\mathbf{d}_\delta$ is a $\mathfrak{u}$-free $(\bar\alpha^\delta, \beta)$-triangle where $\bar\alpha^\delta = \langle \mathbf{f}^{\alpha_\zeta}(\delta) : \zeta \le \varepsilon \rangle, \beta = \varepsilon$ such that $M^{\mathbf{d}_\delta}_{i,\zeta} = M^{\alpha_\zeta}_{\delta+i}$ for $\zeta \le \varepsilon, i \le \mathbf{f}^{\alpha_\zeta}(\delta)$ and $\mathbf{J}^{\mathbf{d}_\delta}_{i,\zeta} = \mathbf{J}^{\alpha_\zeta}_{\delta+i}$ for $\zeta \le \varepsilon, i < \mathbf{f}^{\alpha_\zeta}(\delta)$ and $\mathbf{I}^{\mathbf{d}_\delta}_{i,\zeta} = \emptyset$ for $\zeta < \varepsilon, i \le \mathbf{f}^{\alpha_\zeta}(\delta)$.

Sorting out the definition by 5.14(3A)(a) and the correctness claim 5.19(3)(b), clearly:

$(*)$ for a club of $\gamma < \partial$, if $\gamma \in S$ <u>then</u> $M^{\alpha_\kappa}_{\gamma+\mathbf{f}^{\alpha_\kappa}(\gamma)}$ is brimmed over $M^{\alpha_\kappa}_\gamma$ (and even $M^{\alpha_\kappa}_{\gamma+i}$ for $i < \mathbf{f}^{\alpha_\kappa}(\gamma)$)

which means that $M^\delta_{\gamma+\mathbf{f}^\delta(\gamma)}$ is brimmed over M^δ_γ.

This is clearly enough.

2) Should be clear. $\square_{6.2}$

6.3 Conclusion. For any $(\bar M, \bar{\mathbf{J}}, \mathbf{f}) \in K^{\rm qt}_{\mathfrak{u}}$ and for stationary $S \subseteq \partial$ there is $\mathfrak{u}-2$-appropriate $\mathfrak{g}$ with $S_{\mathfrak{g}} = S$ (see 1.22 and 1.23) such that:

⊛ if $\langle (\bar M^\alpha, \bar{\mathbf{J}}^\alpha, \mathbf{f}^\alpha) : \alpha \le \partial \rangle$ is $\le^{\rm qs}_{\mathfrak{u}}$-increasing continuous and 2-obeys $\mathfrak{g}$ such that $(\bar M^0, \bar{\mathbf{J}}^0, \mathbf{f}^0) = (\bar M, \bar{\mathbf{J}}, \mathbf{f})$ (so $\mathbf{f}^\partial(\delta) = \sup\{\mathbf{f}^\alpha(\delta) : \alpha < \delta\}$ for a club of $\delta < \partial$) <u>then</u> for a club of $\delta < \partial$ the model $M^\partial_\delta = \cup\{M^\alpha_\delta : \alpha < \delta\} \in K_{\mathfrak{s}}$ is brimmed over $M_\delta = M^0_\delta$.

Proof. Similar to the proof of 6.2; only the $\mathfrak{u}$-free triangle is flipped, i.e. it is a dual($\mathfrak{u}$)-free triangle but dual($\mathfrak{u}$) $= \mathfrak{u}$. $\square_{6.3}$

6.4 Definition. Let $1 \le \xi \le \lambda^+$, if we omit it we mean $\xi = \lambda^+$.

1) $K^{3,\rm up}_{\mathfrak{s},\xi}$ is the class of $(M, N, a) \in K^{3,\rm bs}_{\mathfrak{s}}$ which has up-ξ-uniqueness, which means:

⊛ if $(M, N, a) \le^1_{\mathfrak{u}} (M', N', a)$ and $\mathbf{d}$ is a $\mathfrak{u}$-free $(\alpha, 0)$-rectangle with $\alpha \le \xi, \alpha < \lambda^+$ satisfying $(M^{\mathbf{d}}_{0,0}, M^{\mathbf{d}}_{\alpha,0}) = (M, M')$ <u>then</u> $\mathbf{d}$ can be lifted for $((M, N, a), N')$ which means:

⊡ we can find a $\mathfrak{u}$-free $(\alpha+1, 1)$-rectangle $\mathbf{d}^*$ and f such that $\mathbf{d}^* \restriction (\alpha, 0) = \mathbf{d}, f(N) \le_{\mathfrak{s}} M^{\mathbf{d}^*}_{0,1}, f(a) = a^{\mathbf{d}^*}_{0,0}$, i.e. $\mathbf{I}^{\mathbf{d}^*}_{0,0} = \{f(a)\}$ and f is a $\le_{\mathfrak{s}}$-embedding of N' into $M^{\mathbf{d}^*}_{\alpha+1,1}$ over M' hence also $f \restriction M = {\rm id}_M$.

2) We say that $K^{3,\mathrm{up}}_{\mathfrak{s},\xi}$ is dense (in $K^{3,\mathrm{bs}}_{\mathfrak{s}}$) or $\mathfrak{s}$ has density for $K^{3,\mathrm{up}}_{\mathfrak{s},\xi}$ <u>when</u> for every $(M_0,N_0,a) \in K^{3,\mathrm{bs}}_{\mathfrak{s}}$ there is $(M_1,N_1,a) \in K^{3,\mathrm{up}}_{\mathfrak{s},\xi}$ such that $(M_0,N_0,a) \le_{\mathrm{bs}} (M_1,N_1,a)$.
3) We say that $\mathfrak{s}$ has (or satisfies) existence for $K^{3,\mathrm{up}}_{\mathfrak{s},\xi}$ or $K^{3,\mathrm{up}}_{\mathfrak{s},\xi}$ has (or satisfies) existence <u>when</u>: if $M \in K_{\mathfrak{s}}$ and $p \in \mathscr{S}^{\mathrm{bs}}_{\mathfrak{s}}(M)$ <u>then</u> for some pair (N,a) we have $(M,N,a) \in K^{3,\mathrm{up}}_{\mathfrak{s},\xi}$ and $p = \mathbf{tp}_{\mathfrak{s}}(a,M,N)$.
3A) We say that $\mathfrak{s}$ has existence for $K^{3,\mathrm{up}}_{\mathfrak{s},<\xi}$ <u>when</u> $\mathfrak{s}$ has existence for $K^{3,\mathrm{up}}_{\mathfrak{s},\zeta}$ for every $\zeta \in [1,\xi)$; similarly in the cases below.
4) Let $K^{3,\mathrm{up}}_{\mathfrak{s},\xi} = \cap\{K^{3,\mathrm{up}}_{\mathfrak{s},\zeta} : \zeta < \xi\}$.
5) Let $K^{3,\mathrm{up+rd}}_{\mathfrak{s},\xi}$ be defined as $K^{3,\mathrm{up}}_{\mathfrak{s},\xi} \cap K^{3,\mathrm{rd}}_{\mathfrak{s}}$ recalling Definition 5.21, and we repeat parts (2),(3),(4) for it.

6.5 Observation. 1) $K^{3,\mathrm{up}}_{\mathfrak{s},\xi} \subseteq K^{3,\mathrm{up}}_{\mathfrak{s},\zeta} \subseteq K^{3,\mathrm{rd}}_{\mathfrak{s}}$ recalling Definition 5.21 when $1 \le \zeta \le \xi \le \lambda^+$.
2) $\xi = \lambda^+ \Rightarrow K^{3,\mathrm{up}}_{\mathfrak{s},\xi} = \cap\{K^{3,\mathrm{up}}_{\mathfrak{s},\zeta} : \zeta \in [1,\xi)\}$.
3) The triple $(M,N,a) \in K^{3,\mathrm{bs}}_{\mathfrak{s}}$ does not belong to $K^{3,\mathrm{up}}_{\mathfrak{s},\xi}$ <u>iff</u> we can find $\mathbf{d}_1,\mathbf{d}_2$ such that for $\ell = 1,2$

$\square$ (a) $\mathbf{d}_\ell$ is a $\mathfrak{u}$-free $(\alpha_\ell,1)$-rectangle and $\alpha_\ell < \min\{\xi+1,\lambda^+_{\mathfrak{s}}\}$
- (b) $(M^{\mathbf{d}_\ell}_{0,0}, M^{\mathbf{d}_\ell}_{0,1}, \mathbf{I}^{\mathbf{d}_\ell}_{0,0}) = (M,N,\{a\})$
- (c) $M^{\mathbf{d}_1}_{\alpha_1,0} = M^{\mathbf{d}_2}_{\alpha_2,0}$ and $M^{\mathbf{d}_1}_{0,0} = M = M^{\mathbf{d}_2}_{0,0}$
- (d) there is no triple $(\mathbf{d},f)$ such that
  - (α) $\mathbf{d}$ is a $\mathfrak{u}$-free $(\alpha_{\mathbf{d}},\beta_{\mathbf{d}})$-rectangle
  - (β) $\mathbf{d} \restriction (\alpha_1,1) = \mathbf{d}_1$ so $\beta_{\mathbf{d}} \ge 1, \alpha_{\mathbf{d}} \ge \alpha_1$
  - (γ) f is a $\le_{\mathfrak{s}}$-embedding of $M^{\mathbf{d}_2}_{\alpha_2,1}$ into $M^{\mathbf{d}}_{\alpha_{\mathbf{d}},\beta_{\mathbf{d}}}$ over $M^{\mathbf{d}}_{\alpha_1,0} = M^{\mathbf{d}_1}_{\alpha(\mathbf{d}_1),0}$ mapping $M^{\mathbf{d}_2}_{0,1}$ into $M^{\mathbf{d}}_{0,\beta(\mathbf{d})}$ and $a^{\mathbf{d}_2}_{0,0}$ to itself.

Proof. 0) By the definition; as for a $\mathfrak{u}$-free (α,β)-rectangle or $(\bar{\alpha},\beta)$-triangle $\mathbf{d}$ we have: if $i = \min\{i_1,i_2\}, j = \min\{j_1,j_2\}$ and $M^{\mathbf{d}}_{i_1,j_1}, M^{\mathbf{d}}_{i_2,j_2}$ are well defined then $M^{\mathbf{d}}_{i,j} = M^{\mathbf{d}}_{i_1,j_1} \cap M^{\mathbf{d}}_{i_2,j_2}$.
1) By the definition (no need of categoricity).
2) By 5.8(3).
3) Straight, recalling that $\mathfrak{u}$ satisfies monotonicity, (E)(e), see 1.13 but we elaborate.

<u>The Direction $\Rightarrow$</u>:

So (M, N, a) belongs to $K^{3,\mathrm{bs}}_{\mathfrak{s}}$ but not to $K^{3,\mathrm{up}}_{\mathfrak{s},\xi}$. So by Definition 6.4(1) there is $(M', N', \mathbf{d})$ exemplifying the failure of $\circledast$ from 6.4(1), which means

$\odot$ (a) $(M, N, a) \le^1_{\mathfrak{u}} (M', N', a)$, i.e. see Definition 5.10, i.e. 4.29, i.e. $M = N \cap M'$ and $(M, N, a) \le_{\mathrm{bs}} (M', N', a)$

(b) $\mathbf{d}$ is a $\mathfrak{u}$-free $(\alpha, 0)$-rectangle with $\alpha \le \xi, \alpha < \lambda^+$

(c) $(M^{\mathbf{d}}_{0,0}, M^{\mathbf{d}}_{\alpha,0}) = (M, M')$

(d) $\mathbf{d}$ cannot be lifted for (M, N, a, N'), i.e. there is no pair $(\mathbf{d}^*, f)$ such that

(α) $\mathbf{d}^*$ is a $\mathfrak{u}$-free $(\alpha + 1, 1)$-rectangle

(β) $\mathbf{d}^* \restriction (\alpha, 0) = \mathbf{d}$

(γ) f is a $\le_{\mathfrak{s}}$-embedding of N' into $M^{\mathbf{d}^*}_{\alpha+1,1}$ over $M' = M^{\mathbf{d}}_{\alpha,0}$

(δ) $f(N) \le_{\mathfrak{s}} M^{\mathbf{d}^*}_{0,1}$ and $f(a) = a^{\mathbf{d}^*}_{0,0}$.

We define $\mathbf{d}_2$ by

$\odot_2$ $\mathbf{d}_2$ is the $\mathfrak{u}$-free $(1, 1)$-rectangle if

(a) $(M^{\mathbf{d}_2}_{0,0}, M^{\mathbf{d}_2}_{1,0}, \mathbf{J}^{\mathbf{d}_2}_{0,0}) = (M, M', \emptyset)$

(b) $(M^{\mathbf{d}_2}_{0,0}, M^{\mathbf{d}_2}_{1,0}, \mathbf{I}^{\mathbf{d}_2}_{0,0}) = (M, N, \{a\})$

(c) $M^{\mathbf{d}_2}_{1,1} = N, \mathbf{J}^{\mathbf{d}_2}_{0,1} = \emptyset, \mathbf{I}^{\mathbf{d}_2}_{1,0} = \{a\}$.

We choose $\mathbf{d}_1$ such that

$\odot_2$ (a) $\mathbf{d}_2$ is a $\mathfrak{u}$-free $(\alpha, 1)$-rectangle

(b) $\mathbf{d}_2 \restriction (\alpha, 0) = \mathbf{d}^*$

(c) $(M^{\mathbf{d}_2}_{0,0}, M^{\mathbf{d}_2}_{0,1}, \mathbf{I}^{\mathbf{d}_2}_{0,0}) = (M, N, \{a\})$.

<u>The Direction $\Leftarrow$</u>:

Choose $\mathbf{d} = \mathbf{d}_1$ and use Exercise 1.13. $\square_{6.5}$

6.6 Definition. We say (M, N, a) is a up-orthogonal to $\mathbf{d}$ <u>when</u>:

$\circledast$ (a) $(M, N, a) \in K^{3,\mathrm{bs}}_{\mathfrak{s}}$

(b) $\mathbf{d}$ is a $\mathfrak{u}$-free $(\alpha_{\mathbf{d}}, 0)$-rectangle

(c) $M^{\mathbf{d}}_{0,0} = M$

(d) <u>Case 1</u>: $N \cap M^{\mathbf{d}}_{\alpha(\mathbf{d}),0} = M$.

If N_1 satisfies $(M, N, a) \leq^1_{\mathfrak{u}} (M^{\mathbf{d}}_{\alpha(\mathbf{d}),0}, N_1, a)$,
so N_1 does $\leq_{\mathfrak{s}}$-extends $M^{\mathbf{d}}_{\alpha(\mathbf{d}),0}$ and N, <u>then</u> the rectangle $\mathbf{d}$ can be lifted for $((M, N, a), N_1)$;
<u>Case 2</u>: possibly $N \cap M^{\mathbf{d}}_{(\alpha(\mathbf{d}),0)} \neq M$.

We replace (N, a) by (N', a')
such that $N' \cap M^{\mathbf{d}}_{\alpha(\mathbf{d}),0} = M$ and there is an isomorphism from N onto N' over M mapping a to a'.

We now consider a relative of Definition 6.4.

6.7 Definition. Let $\xi \leq \lambda^+$ but $\xi \geq 1$, if $\xi = \lambda^+$ we may omit it.
1) We say that $\mathfrak{s}$ has almost-existence for $K^{3,\mathrm{up}}_{\mathfrak{s},\xi}$ <u>when</u>: if $M \in K_{\mathfrak{s}}$ and $p \in \mathscr{S}^{\mathrm{bs}}_{\mathfrak{s}}(M)$ we have

$\odot_{M,p}$ if $\alpha \leq \xi$ and $\mathbf{d}$ is a $\mathfrak{u}$-free $(\alpha, 0)$-rectangle with $M^{\mathbf{d}}_{0,0} = M$ (yes, we allow $\alpha = \xi = \lambda^+$) <u>then</u> there is a triple $(M, N, a) \in K^{3,\mathrm{bs}}_{\mathfrak{s}}$ such that $p = \mathbf{tp}_{\mathfrak{s}}(a, M, N)$ and (M, N, a) is up-orthogonal to $\mathbf{d}$.

2) We say that $\mathfrak{s}$ has the weak density for $K^{3,\mathrm{up}}_{\mathfrak{s},\xi}$ <u>when</u>: if $M \in K_{\mathfrak{s}}$ and $p \in \mathscr{S}^{\mathrm{bs}}_{\mathfrak{s}}(M)$ <u>then</u> for some (M_1, p_1) the demand $\odot_{M_1,p_1}$ in part (1) holds and $M \leq_{\mathfrak{s}} M_1$ and $p_1 \in \mathscr{S}^{\mathrm{bs}}_{\mathfrak{s}}(M_1)$ is a non-forking extension of p.
3) We write "almost-existence/weak density for $K^{3,\mathrm{up}}_{\mathfrak{s},<\xi}$" when this holds for every $\xi' < \xi$.

6.8 Observation. Assume $\mathfrak{s}$ is categorical (in $\lambda_{\mathfrak{s}}$) and $\xi \leq \lambda^+$.
1) Then $\mathfrak{s}$ has weak density for $K^{3,\mathrm{up}}_{\mathfrak{s},\xi}$ <u>iff</u> $\mathfrak{s}$ has almost existence for $K^{3,\mathrm{up}}_{\mathfrak{s},\xi}$.
2) If $\mathfrak{s}$ has existence for $K^{3,\mathrm{up}}_{\mathfrak{s},\xi}$ <u>then</u> $\mathfrak{s}$ has almost existence for $K^{3,\mathrm{up}}_{\mathfrak{s},\xi}$.

Proof. 1) The weak density version implies the existence version, i.e. the first implies the second because if $M \leq_{\mathfrak{s}} M_1$ and $p_1 \in \mathscr{S}^{\mathrm{bs}}_{\mathfrak{s}}(M_1)$ does not fork over M then there is an isomorphism f from M_1 onto M mapping p to $p \restriction M$, see 5.20.

The inverse is obvious.
2) Read the definitions. $\square_{6.8}$

6.9 Discussion: Below we fix $p_* \in \mathscr{S}^{\text{bs}}_{\mathfrak{s}}(M_*)$ and look only at stationarization of p_*. We shall use the failure of almost-existence for $K^{3,\text{up}}_{\mathfrak{s},\xi}$ to get non-structure.

We first present a proof in the case $\mathscr{D}_\partial$ is not ∂^+-saturated (see 6.15) but by a more complicated proof this is not necessary, see 6.14. As it happens, we do not assume $2^\lambda < 2^{\lambda^+}$, but still assume $2^{\lambda^+} < 2^{\lambda^{++}}$ using the failure of weak density for $K^{3,\text{up}}_{\mathfrak{s},<\lambda^+}$ to get an up-invariant coding property (avoiding the problem we encounter when we try to use $\mathbf{d}_\delta$ depending on N^η_δ).

So in 6.15 for each $\alpha < \lambda^{++} = \partial^+$ we "give" a stationary $S_\alpha \subseteq \partial$ almost disjoint to S_β for $\beta < \alpha^+$.

Well, we have for the time being decided to deal only with up-uniqueness arguing that it will help to deal with "true" uniqueness. Also, in the non-structure we use failure of weak density for $K^{3,\text{up}}_{\mathfrak{s},\xi}$, whereas for the positive side, in §7, we use existence. The difference is that for existence we have (N,a) for given (M,p,ξ) whereas for almost existence we are given $(M,p,\mathbf{d})$. However, we now prove their equivalence. To get the full theorem 6.14 we use 3.10 - 3.24.

6.10 Claim. *Assume $\mathfrak{s}$ is categorical; if $\xi \le \lambda^+$ and $\mathfrak{s}$ has almost-existence for $K^{3,\text{up}}_{\mathfrak{s},\xi}$ then $\mathfrak{s}$ has existence for $K^{3,\text{up}}_{\mathfrak{s},\xi}$.*

Proof. This follows from the following two subclaims, 6.11, 6.12.

6.11 Subclaim. *If $(M,N,a) \in K^{3,\text{bs}}_{\mathfrak{s}}$ is up-orthogonal to $\mathbf{d}_2$, then it is up-orthogonal to $\mathbf{d}_1$ when:*

$\circledast$ (a) $\mathbf{d}_\ell$ *is a $\mathfrak{u}$-free $(\alpha_\ell,0)$-rectangle for $\ell = 1,2$ where $\alpha_\ell \le \lambda^+$*

(b) $M^{\mathbf{d}_1}_{0,0} = M = M^{\mathbf{d}_2}_{0,0}$

(c) *h is an increasing function from α_1 to α_2*

(d) *f is an $\le_{\mathfrak{s}}$-embedding of $M^{\mathbf{d}_1}_{\alpha_1,0}$ into $M^{\mathbf{d}_2}_{\alpha_2,0}$*

(e) $f \restriction M^{\mathbf{d}_1}_{0,0} = \text{id}_M$

(f) *if $\beta < \alpha_1$ then*

(α) $f(b^{\mathbf{d}_1}_{\beta,0}) = b^{\mathbf{d}_2}_{h(\beta),0}$

(β) *f maps $M^{\mathbf{d}_1}_{\beta,0}$ into $M^{\mathbf{d}_2}_{h(\beta),0}$*

(γ) $\mathbf{tp}_{\mathfrak{s}}(b^{\mathbf{d}_2}_{h(\beta),0}, M^{\mathbf{d}_2}_{h(\beta),0}, M^{\mathbf{d}_2}_{h(\beta)+1,0})$ *does not fork over $f(M^{\mathbf{d}_1}_{\beta,0})$.*

Proof. Without loss of generality f is the identity and $M^{\mathbf{d}_2}_{\alpha_2,0} \cap N = M$.

So assume $N_1 \in K_{\mathfrak{s}}$ is a $\le_{\mathfrak{s}}$-extension of N and of $M^{\mathbf{d}_1}_{\alpha_1,0}$ such that $(M,N,a) \le_{\rm bs} (M^{\mathbf{d}_1}_{\alpha_1,0}, N_1, a)$ and we should prove the existence of a suitable lifting. Without loss of generality $N_1 \cap M^{\mathbf{d}_2}_{\alpha_2,0} = M^{\mathbf{d}_1}_{\alpha_1,0}$. Hence there is N_2 which does $\le_{\mathfrak{s}}$-extend $M^{\mathbf{d}_2}_{\alpha_2,0}$ and N_1 and $(M^{\mathbf{d}_1}_{\alpha_1,0}, N_1, a) \le^1_{\mathfrak{u}} (M^{\mathbf{d}_2}_{\alpha_2,0}, N_2, a)$; but $\le^1_{\mathfrak{u}}$ is a partial order hence $(M,N,a) \le^1_{\mathfrak{u}} (M^{\mathbf{d}_2}_{\alpha_{2,0}}, N_2, a)$.

Recall that we are assuming (M,N,a) is up-orthogonal to $\mathbf{d}_2$ hence we can find $\mathbf{d}^2, f$ as in Definition 6.6, i.e. as in $\odot$ inside Definition 6.4(1), so $\mathbf{d}^2$ is a $\mathfrak{u}$-free $(\alpha_2+1,1)$-rectangle, $\mathbf{d}^2 \restriction (\alpha_2,0) = \mathbf{d}_2, f$ is a $\le_{\mathfrak{s}}$-embedding of N_2 into $M^{\mathbf{d}_2}_{\alpha_2+1,1}$ over $M^{\mathbf{d}_2}_{\alpha_2,0}$ mapping N into $M^{\mathbf{d}}_{0,1}$ satisfying $f(a) = a^{\mathbf{d}^2}_{0,0}$, note: $\mathbf{I}^{\mathbf{d}}_{\alpha,0} = \{a\}$ for $\alpha \le \alpha_2+1$. Now we define $\mathbf{d}^1$, a $\mathfrak{u}$-free $(\alpha_1+1,1)$-rectangle by

$\boxtimes$ (a) $\mathbf{d}^1 \restriction (\alpha_1,0) = \mathbf{d}_1$

(b) $M^{\mathbf{d}^1}_{\alpha,1}$ is $M^{\mathbf{d}^2}_{h(\alpha),1}$ if $\alpha \le \alpha_1$ is a non-limit ordinal and is $\cup\{M^{\mathbf{d}^2}_{\beta,1} : \beta < h(\alpha)\}$ if $\alpha \le \alpha_1$ is a limit ordinal and is $M^{\mathbf{d}^2}_{\alpha_1+1,1}$ if $\alpha = \alpha_1+1$

(c) $M^{\mathbf{d}^1}_{\alpha_1+1,0} = M^{\mathbf{d}^2}_{\alpha_1,0}$

(d) $\mathbf{I}^{\mathbf{d}_1}_{\alpha,0} = \mathbf{I}^{\mathbf{d}_2}_{h(\alpha),0}$ for $\alpha \le \alpha_1$

(e) $\mathbf{J}^{\mathbf{d}_1}_{\alpha,1} = \mathbf{J}^{\mathbf{d}_2}_{h(\alpha),0}$ for $\alpha < \alpha_1$

(f) $\mathbf{I}^{\mathbf{d}^1}_{\alpha_1+1,0} = \mathbf{I}^{\mathbf{d}^2}_{\alpha_2+1,0}$

(g) $\mathbf{J}^{\mathbf{d}^1}_{\alpha_1,0} = \emptyset = \mathbf{J}^{\mathbf{d}^1}_{\alpha_1,1}$.

Now check. $\square_{6.11}$

6.12 Claim. *For every $\alpha_1 \le \lambda^+$ there is $\alpha_2 \le \lambda^+$ (in fact $\alpha_2 = \lambda\alpha_1$ is O.K.) such that: for every $M \in K_{\mathfrak{s}}$ there is a $\mathfrak{u}$-free $(\alpha_2,0)$-rectangle $\mathbf{d}_2$ with $M^{\mathbf{d}_2}_{\alpha_2,0} = M$ such that*

(∗) *if $\mathbf{d}_1$ is a $\mathfrak{u}$-free $(\alpha_1,0)$-rectangle with $M^{\mathbf{d}_1}_{0,0} = M$ <u>then</u> there are h,f as in $\circledast$ of 6.11.*

Proof. Let $\langle \mathscr{U}_i : i \le \lambda\rangle$ be a $\subseteq$-increasing continuous sequence of subsets of λ such that $\mathscr{U}_\lambda = \lambda$, $\min(\mathscr{U}_i) \ge i, \lambda = |\mathscr{U}_0|$ and $\lambda = |\mathscr{U}_{\alpha+1} \backslash \mathscr{U}_\alpha|$ for $\alpha < \lambda$. We now choose $(M_i, \bar{p}_i, a_i)$ by induction on $i \le \lambda\alpha_1$ such that

$\oplus$ (a) $\langle M_i : j \le i\rangle$ is $\le_{\mathfrak{s}}$-increasing continuous

(b) $M_0 = M$

(c) if $i = j+1$ then M_i is $\le_{\mathfrak{s}}$-universal over M_j

(d) if $i = \lambda i_1 + i_2$ and $i_1 < \alpha_1, i_2 < \lambda$ then $\bar p_i = \langle p^{i_1}_\varepsilon : \varepsilon \in \mathscr{U}_{i_2}\rangle$ where $p^{i_1}_\varepsilon \in \cup \mathscr{S}^{\rm bs}_{\mathfrak{s}}(M_{\lambda i_1 + i}) : i \le i_2\}$

(e) if $i = \lambda i_1 + i_2, i_1 < \alpha_1, j \le i_2 < \lambda$ and $q \in \mathscr{S}^{\rm bs}_{\mathfrak{s}}(M_{\lambda i_1 + i_2})$ then $(\exists^\lambda \zeta \in \mathscr{U}_{i_2})(p^i_\zeta = q)$

(f) if $i = \lambda i_1 + i_2, i_2 = j_2 + 1 < \lambda$, and $j_2 \in \mathscr{U}_\varepsilon$ then $a_{i-1} \in M_i$ and the type $\mathbf{tp}_{\mathfrak{s}}(a_{i-1}, M_{i-1}, M_i)$ is a non-forking extension of $p^{i_1}_{j_2}$.

Now choose $\mathbf{d}_2 = (\langle M_i : i \le \lambda\alpha\rangle, \langle a_i : i < \lambda\alpha\rangle)$. So assume $\mathbf{d}_1$ is a $\mathfrak{u}$-free $(\alpha_1, 0)$-rectangle with $M^{\mathbf{d}_1}_{0,0} = M = M^{\mathbf{d}_2}_{0,0}$. We now choose a pair (f_i, h_i) by induction on $i \le \alpha_1$ such that: f_i is a $\le_{\mathfrak{s}}$-embedding of $M^{\mathbf{d}_1}_i$ into $M^{\mathbf{d}_2}_{\lambda,i}, h_i : i \to \lambda i$ such that $h(j) \in (\lambda j, \lambda + \lambda), f_i$ is $\subseteq$-increasing, h_i is $\subseteq$-increasing, $\mathbf{tp}_{\mathfrak{s}}(a^{\mathbf{d}_2}_{h(j),0}, M^{\mathbf{d}_2}_{h(j)+1,0})$ is a non-forking extension of $f_{j+1}(\mathbf{tp}_{\mathfrak{s}}(a^{\mathbf{d}_1}_j, M^{\mathbf{d}_1}_j, M^{\mathbf{d}_1}_{j+1}))$ and check; in fact $\oplus$ gives more than necessary. $\square_{6.12}$ $\square_{6.10}$

6.13 Claim. *Assume $\xi \le \lambda^+$ and $M_* \in K_{\mathfrak{s}}$ and $p_* \in \mathscr{S}^{\rm bs}_{\mathfrak{s}}(M_*)$ witness that $\mathfrak{s}$ fails the weak density for $K^{3,\rm up}_{\mathfrak{s},\xi}$, see Definition 6.7.*
1) If $(M, N, \{a\}) \in K^{3,\rm bs}_{\mathfrak{s}}$ and $M_* \le_{\mathfrak{s}} M$ and $\mathbf{tp}_{\mathfrak{s}}(a, M, N)$ is a non-forking extension of p_* <u>then</u> $(M, N, \{a\})$ has the weak ξ-uq-invariant coding property for $\mathfrak{u}$; pedantically assuming $\mathfrak{s}$ has fake equality see Definition 4.25, see 6.18, similarly in part (2); on this coding property, (see Definition 3.2(1).
2) Moreover the triple $(M, N, \{a\}) \in {\rm FR}^1_{\mathfrak{u}}$ has the semi ξ-uq-invariant coding property, (see Definition 3.14).

Remark. If below 6.15 suffices for us then part (2) of 6.13 is irrelevant.

Proof. 1) Read the definitions, i.e. Definition 6.7(2) on the one hand and Definition 3.2(1) on the other hand. Pedantically one may worry that in 6.7(2) we use $\le^{\rm bs}_{\mathfrak{s}}$, where disjointness is not required whereas in $\le^\ell_{\mathfrak{u}}$ it is, however as we allow using fake equality in $\mathfrak{K}_{\mathfrak{u}}$ this is not problematic.
2) Similar. $\square_{6.13}$

Now we arrive to the main result of the section.

6.14 Theorem. *$\dot I(\lambda^{++},\mathfrak{K}^{\mathfrak{s}}) \geq \mu_{\rm unif}(\lambda^{++},2^{\lambda^+})$ and even $\dot I(\lambda^{++},K^{\mathfrak{s}}(\lambda^+$-saturated above $\lambda)) \geq \mu_{\rm unif}(\lambda^{++},2^{\lambda^+})$ and even $\dot I(K^{\mathfrak{s},\mathfrak{h}}_{\lambda^{++}}) \geq \mu_{\rm unif}(\lambda^{++},2^{\lambda^+})$ for any $\mathfrak{u}_{\mathfrak{s}} - \{0,2\}$-appropriate $\mathfrak{h}$ <u>when</u>:*

(a) $2^{\lambda^+} < 2^{\lambda^{++}}$

(b) (α) $\mathfrak{s}$ fail the weak density for $K^{3,\rm up}_{\mathfrak{s},\xi}$ where $\xi \leq \lambda^+$ or just for $\xi = \lambda^+$

(β) if $\xi = \lambda^+$ then $2^\lambda < 2^{\lambda^+}$.

Before we prove 6.14 we prove a weaker variant when we strengthen the set theoretic assumption.

6.15 Theorem. *Like 6.14 but we add to the assumption*

(c) $\mathscr{D}_{\lambda^+}$ is not λ^{++}-saturated (see 6.16(1) below).

6.16 Remark. 0) In the section main conclusion, 6.17, if we add clause (c) of 6.15 to the assumptions then we can rely there on 6.15 instead of on 6.14.
1) Recall that $\lambda > \aleph_0 \Rightarrow \mathscr{D}_{\lambda^+}$ is not λ^+-saturated by Gitik-Shelah [GiSh 577], hence this extra set theoretic assumption is quite weak.
2) We use 3.5 in proving 6.15.
3) We can add the version with $\mathfrak{h}$ to the other such theorems.

Proof of 6.15. We can choose a stationary $S \subseteq \partial = \lambda^+$ such that $\mathscr{D}_{\lambda^+} + (\lambda^+ \backslash S)$ is not λ^{++}-saturated. We shall apply Theorem 3.5 for the S we just chose for ξ' which is $\lambda+1$ if $\xi = \lambda$ and is ξ if $\xi < \lambda^+$.

We have to verify 3.5's assumption (recalling $\partial = \lambda^+$): clauses (a) + (b) of 3.5 holds by clauses (a) + (c) of the assumption of 6.15 if $\xi < \lambda^+$ and clauses (a) + (b)(β) of the assumptions of 6.15 if $\xi = \lambda^+$. Clause (c) of 3.5 holds by 6.2, 6.13(1), whose assumption holds by clause (b) of the assumption of 6.14. Really we have to use 6.13(1), 1.8(6). $\square_{6.14}$

6.17 Conclusion. Assume $2^{\lambda^+} < 2^{\lambda^{++}}$ and $\xi \leq \lambda^+$ but $\xi = \lambda^+ \Rightarrow 2^\lambda < 2^{\lambda^+}$.
1) If $\dot I(\lambda^{++},K^{\mathfrak{s}}(\lambda^+$-saturated)) is $< \mu_{\rm unif}(\lambda^{++},2^{\lambda^+})$ and $\mathfrak{K}_{\mathfrak{s}}$ is categorical <u>then</u> $\mathfrak{s}$ has existence for $K^{3,\rm up}_{\mathfrak{s},\xi}$ for every $\xi \leq \lambda^+$.
2) Similarly for $\dot I(\mathfrak{K}^{\mathfrak{s},\mathfrak{h}}_{\lambda^{++}})$ for any $\mathfrak{u}_{\mathfrak{s}} - \{0,2\}$-appropriate $\mathfrak{h}$.

Proof. Let $\xi \leq \lambda^+$. We first try to apply Theorem 6.14. Its conclusion fails, but among its assumptions clauses (a) and (b)(β) hold by our present assumptions. So

necessarily the demand in clause (b)(α) of 6.14 fails. So we have deduced that $\mathfrak{s}$ has weak density for $K^{3,\mathrm{up}}_{\mathfrak{s},\xi}$. By Observation 6.8(1), recalling we are assuming that $\mathfrak{K}_{\mathfrak{s}}$ is categorical, it follows that $\mathfrak{s}$ has almost existence for $K^{3,\mathrm{up}}_{\mathfrak{s},\xi}$. By Claim 6.10 again recalling we are assuming $\mathfrak{K}_{\mathfrak{s}}$ is categorical we can deduce that $\mathfrak{s}$ has existence for $K^{3,\mathrm{up}}_{\mathfrak{s},\xi}$.

So we have gotten the desired conclusion. But we still have to prove Theorem 6.14 in the general case. $\square_{6.17}$

6.18 Claim. *Assume $\mathfrak{t}'$ is an almost good λ-frame derived from $\mathfrak{t}$ as in Definition 4.25 and $\mathfrak{u}' = \mathfrak{u}^1_{\mathfrak{t}'}$ is as defined in Definition 5.10, i.e. Definition 4.29 (and see Claims 4.30, 5.11).*

<u>*Then*</u>

$\boxtimes$ (a) $\mathfrak{t}'$ *satisfies all the assumptions on* $\mathfrak{t}$ *in 6.1*

(b) *so all that we have proved on* $(\mathfrak{t},\mathfrak{u})$ *in this section apply to* $(\mathfrak{t}',\mathfrak{u}')$*, too*

(c) $\mathfrak{u}'$ *has fake equality* $=_\tau$ *(see Definition 3.17(1)*

(d) $\mathfrak{u}'$ *is hereditary for the fake equality* $=_\tau$ *(see Definition 3.17(4)).*

Proof. Clause (a) holds by Claim 4.26. Clause (b) holds by Claim 4.30, 5.11. Clause (c) holds by direct inspection on 4.29(6)(γ).

In clause (d), "$\mathfrak{u}'$ is hereditary for the fake equality $=_\tau$", i.e. satisfies clause (a) of Definition 3.17(3) by Claim 4.30(6)(β), 5.11 applied to $\mathfrak{t}'$.

Lastly, to prove "$\mathfrak{u}'$ is hereditary for the fake equality $=_\tau$" we have to show that it satisfies clause (b) of 3.17(4), which holds by 4.30(6)(δ). $\square_{6.18}$

<u>6.19 Proof of 6.14</u>: We shall use Claim 6.18 to derive $(\mathfrak{s}',\mathfrak{u}')$, so we can use the results of this section to $(\mathfrak{s},\mathfrak{u})$ and to $(\mathfrak{s}',\mathfrak{u}')$. Now by 6.2 for some $\mathfrak{u}_{\mathfrak{s}'}-2$-appropriate $\mathfrak{h}$, every $M \in K^{\mathfrak{u}',\mathfrak{h}}_{\partial^+}$ is τ-fuller, see Definition 1.8(6), so by 1.8(6) it is enough to prove Theorem 6.14 for $(\mathfrak{s}',\mathfrak{u}')$. Now by Claim 6.13(1) and clause (b)(α) of the assumption we know that some $(M,N,\{a\}) \in \mathrm{FR}^1_{\mathfrak{u}'}$ has the semi uq-invariant coding property (for $\mathfrak{u}'$). Also $\mathfrak{u}'$ has the fake equality $=_\tau$ and is hereditary for it by 6.18 and is self dual by 5.11(1).

Hence in Claim 3.20 all the assumptions hold for $\mathfrak{u}',(M,N,a)$, hence its conclusion holds, i.e. $(M,N,\{a\})$ has the weak vertical ξ-uq-invariant coding property. This means that clause (b) from the assumptions of Theorem 3.24 holds. Clause (a) there means $2^\lambda < 2^{\lambda^+} < 2^{\lambda^{++}}$ (choosing $\theta := \lambda, \partial = \lambda^+$) and clause (c) holds by 6.2. So we are done. Having shown that the assumptions of Theorem 3.24 hold, we get its conclusion, which is the conclusion of the present theorem (reclaling we show that it suffices to prove it for $\mathfrak{s}'$, so we are done. $\square_{6.14}$

§7 Pseudo uniqueness

Our explicit main aim is to help in §8 to show that under the assumptions of Chapter VI, i.e. [Sh 576] we can get a good λ-frame not just a good λ^+-frame as done in III§3(D),3.7. For this we deal with almost good frames (see 7.1 and Definition 5.2) and assume existence for $K^{3,\mathrm{up}}_{\mathfrak{s},\lambda^+}$ (see Definition 6.4(3A) justified by 6.17) and get enough of the results of III§6 and few from Chapter IV. This means that $\mathrm{WNF}_{\mathfrak{s}}$ is defined in 7.3 and proved to be so called "a weak non-forking relation on $\mathfrak{K}_{\mathfrak{s}}$ respecting $\mathfrak{s}$"; we also look at almost good λ-frames with such relations and then prove that they are good λ-frames in 7.19(2). Those results are used in the proof of 4.32 (and in §8).

But this has also interest in itself as in general we like to understand pre-λ-frames which are not as good as the ones considered in Chapter IV, i.e. weakly successful good λ-frames. We will try to comment on this, too. Note that below even if $\mathfrak{s}$ is a good λ-frame satisfying Hypothesis 7.1 which is weakly successful (i.e. we have existence for $K^{3,\mathrm{uq}}_{\mathfrak{s}}$, still $\mathrm{WNF}_{\mathfrak{s}}$ defined below is not in general equal to $\mathrm{NF}_{\mathfrak{s}}$). We may wonder, is the assumption (3) of 7.1 necessary? The problem is in 7.14, 7.11.

Till 7.20 we use:

7.1 Hypothesis. 1) $\mathfrak{s}$ is an almost good λ-frame (see Definition 5.2).
2) $\mathfrak{s}$ has existence[27] for $K^{3,\mathrm{up}}_{\mathfrak{s},\lambda^+}$, see Definition 6.4(3A) and sufficient condition in 6.17.
3) $\mathfrak{s}$ is categorical in λ (used only from 7.14 on).
4) $\mathfrak{s}$ has disjointness (see Definition 5.5; used only from 7.10 on, just for transparency, in fact follows from parts (1) + (3) by 5.23).

7.2 Definition. 1) Let $\mathfrak{u} = \mathfrak{u}_{\mathfrak{s}}$ be as in Definition 4.29 and 5.10.
2) Let $\mathrm{FR}_{\mathfrak{s}}$ be $\mathrm{FR}^{\ell}_{\mathfrak{u}_{\mathfrak{s}}}$ for $\ell = 1, 2$ (they are equal).

7.3 Definition. 1) Assume $\xi \leq \lambda^+$. Let $\mathrm{WNF}^{\xi}_{\mathfrak{s}}(M_0, N_0, M_1, N_1)$ mean that: $M_0 \leq_{\mathfrak{s}} N_0 \leq_{\mathfrak{s}} N_1, M_0 \leq_{\mathfrak{s}} M_1 \leq_{\mathfrak{s}} N_1$ and if $\alpha < \xi$ so $\alpha < \lambda^+$ and $\mathbf{d}$ is an $\mathfrak{u}$-free $(0, \alpha)$-rectangle and f is a $\leq_{\mathfrak{s}}$-embedding of N_0 into $M^{\mathbf{d}}_{0,\alpha}$ such that $f(M_0) = M^{\mathbf{d}}_{0,0}$ <u>then</u> we can find a model N^* and a $\mathfrak{u}$-free $(1, \alpha)$-rectangle $\mathbf{d}^+$ satisfying $\mathbf{d}^+ \restriction (0, \alpha) = \mathbf{d}$ and $M^{\mathbf{d}^+}_{1,\alpha} \leq_{\mathfrak{s}} N^*$ and $\leq_{\mathfrak{s}}$-embedding $g \supseteq f$ of N_1 into N^* such that $M^{\mathbf{d}^+}_{1,0} = g(M_1)$.
2) If $\xi = \lambda^+$ we may omit it. So $\mathrm{WNF}^{\xi}_{\mathfrak{s}}$ is also considered as the class of such quadruples of models.

[27] if $\mathfrak{s}$ is a good λ-frame, then actually $K^{3,\mathrm{up}}_{\mathfrak{s},<\lambda^+}$ is enough, see 6.4(3A); the main difference is in the proof of 7.14

7.4 Remark. 0) Definition 7.3(1) is dull for $\xi = 0$.
1) So this definition is not obviously symmetric but later we shall prove it is.
2) Similarly, it seemed that the value of ξ is important, but we shall show that for $\xi < \lambda^+$ large enough it is not <u>when</u> $\mathfrak{s}$ is a good λ-frame; see e.g. 7.6.
3) In Definition 7.3 we may ignore $\xi = 0$ as it essentially says nothing.

7.5 Observation. 1) If $1 \le \xi \le \lambda^+$ and $\mathrm{WNF}^\xi_{\mathfrak{s}}(M_0, N_0, M_1, N_1)$ and $(M_0, N_0, a) \in K^{3,\mathrm{bs}}_{\mathfrak{s}}$ <u>then</u> $(M_0, N_0, a) \le_{\mathrm{bs}} (M_1, N_1, a)$ and in particular $(M_1, N_1, a) \in K^{3,\mathrm{bs}}_{\mathfrak{s}}$.
2) $\mathrm{WNF}^\xi_{\mathfrak{s}}$ is $\subseteq$-decreasing with ξ.
3) $\mathrm{WNF}_{\mathfrak{s}} = \mathrm{WNF}^{\lambda^+}_{\mathfrak{s}} = \cap\{\mathrm{WNF}^\xi_{\mathfrak{s}} : \xi < \lambda^+\}$.
4) In Definition 7.3(1) in the end we can weaken "$M^{\mathbf{d}^+}_{1,0} = g(M_1)$" to "$g(M_1) \le_{\mathfrak{s}} M^{\mathbf{d}^+}_{1,0}$".

Proof. 1) Straight, use $\mathbf{d}$, the $\mathfrak{u}$-free $(0,1)$-rectangle such that $M^{\mathbf{d}}_{0,0} = M_0$ and $M^{\mathbf{d}}_{0,1} = M$ and $a^{\mathbf{d}}_{0,0} = a$, i.e. $\mathbf{I}^{\mathbf{d}}_{0,0} = \{a\}$.
2),3) Trivial.
4) Given $(N^*, \mathbf{d}^1, g)$ as in Definition 7.5(4). We define $\mathbf{d}'$, a $\mathfrak{u}$-free $(1,\alpha)$-rectangle by $M^{\mathbf{d}'}_{i,j}$ is $g(M_1)$ if $(i,j) = (1,0)$ and is $M^{\mathbf{d}^+}_{i,j}$ when $i \le 1$ & $j \le \alpha$ & $(i,j) \ne (1,0)$ and $\mathbf{I}^{\mathbf{d}'}_{0,j} = \mathbf{I}^{\mathbf{d}'}_{1,j} = \mathbf{I}^{\mathbf{d}}_{0,j}$ for $j < \alpha$ and $\mathbf{J}^{\mathbf{d}'}_{0,j} = \emptyset$ for $j \le \alpha$. The only non-obvious point is why $(M^{\mathbf{d}'}_{0,0}, M^{\mathbf{d}'}_{0,1}, \mathbf{I}^{\mathbf{d}'}_{0,0}) \le^1_{\mathfrak{u}} (M^{\mathbf{d}'}_{1,0}, M^{\mathbf{d}'}_{1,1}, \mathbf{I}^{\mathbf{d}'}_{0,0})$ which means $(M^{\mathbf{d}}_{0,0}, M^{\mathbf{d}}_{0,1}, \mathbf{I}^{\mathbf{d}}_{0,0}) \le^1_{\mathfrak{u}} (g(M_1), M^{\mathbf{d}^+}_{1,1}, \mathbf{I}^{\mathbf{d}^+}_{0,0})$. This is because $\mathfrak{u}$ is interpolative by 4.30(6)(ε), see Definition 3.21. $\square_{7.5}$

7.6 Claim. *[Monotonicity] Assume* $1 \le \xi \le \lambda^+$. *If* $\mathrm{WNF}^\xi_{\mathfrak{s}}(M_0, N_0, M_1, N_1)$ *and* $M_0 \le_{\mathfrak{s}} N'_0 \le_{\mathfrak{s}} N_0$ *and* $M_0 \le_{\mathfrak{s}} M'_1 \le M_1$ *and* $N_1 \le_{\mathfrak{s}} N'_1, N'_0 \cup M'_1 \subseteq N''_1 \le_{\mathfrak{s}} N'_1$ <u>*then*</u> $\mathrm{WNF}^\xi_{\mathfrak{s}}(M_0, N'_0, M'_1, N''_1)$ *holds.*

Proof. It is enough to prove that for the case three of the equalities $N'_0 = N_0, M'_1 = M_1, N''_1 = N'_1, N'_1 = N_1$ hold. Each follows: in the case $N'_0 \ne N_0$ by the Definition 7.3, in the case $M'_1 \ne M_1$ by 7.5(4), and in the cases $N''_1 = N'_1 \vee N'_1 = N_1$ by amalgamation (in $\mathfrak{K}_{\mathfrak{s}}$) and the definition of $\mathrm{WNF}^\xi_{\mathfrak{s}}$ and 7.5(4).

$\square_{7.6}$

7.7 Observation. [$\mathfrak{s}$ categorical in λ or M_0 is brimmed or $\mathfrak{s}$ has existence for $K^{3,\mathrm{rd}}_{\mathfrak{s}}$, see Definition 5.21.]

If $\xi > \xi^{\mathrm{rd}}_{M_0}$, see Definition 5.21 and $\mathrm{WNF}^\xi_{\mathfrak{s}}(M_0, N_0, M_1, N_1)$ <u>then</u> $M_1 \cap N_0 = M_0$.

Remark. 1) Recall that if $\mathfrak{s}$ is an almost good λ-frame <u>then</u> it has density of $K^{3,\mathrm{rd}}_{\mathfrak{s}}$ hence if $\mathfrak{s}$ is categorical then it also has existence for $K^{3,\mathrm{rd}}_{\mathfrak{s}}$.

It is convenient to assume this but not essential. Proving density for $K^{3,\mathrm{up}}_{\mathfrak{s}}$ we actualy prove density for $K^{3,\mathrm{up}}_{\mathfrak{s}} \cap K^{3,\mathrm{rd}}_{\mathfrak{s}}$; moreover $K^{3,\mathrm{up}}_{\mathfrak{s}} \subseteq K^{3,\mathrm{rd}}_{\mathfrak{s}}$.
2) Recall $\xi^{\mathrm{rd}}_{M_0} = \xi^{\mathrm{rd}}_{\mathfrak{s}}$ if M_0 is superlimit, e.g. when $K_{\mathfrak{s}}$ is categorical.

Proof. Recall that letting $\alpha = \xi^{\mathrm{rd}}_{M_0}$ by 5.22(3) or 5.22(4) there is a $\mathfrak{u}$-free $(0,\alpha)$-rectangle $\mathbf{d}$ such that $M^{\mathbf{d}}_{0,0} = M_0, N_0 \le_{\mathfrak{s}} M^{\mathbf{d}}_{0,\alpha}$ and each $(M^{\mathbf{d}}_{0,\alpha}, M^{\mathbf{d}}_{0,\alpha+1}, a^{\mathbf{d}}_{0,\alpha})$ is reduced (see Definition 5.21). Now apply Definition 7.3 to this $\mathbf{d}$. Alternatively, recall for $\mathfrak{u}$-free (α,β)-rectangle (or $(\bar{\alpha},\beta)$-triangle) $\mathbf{d}$ we have $M^{\mathbf{d}}_{i_1,j_1} \cap M^{\mathbf{d}}_{i_2,j_2} = M^{\mathbf{d}}_{\min\{i_1,i_2\},\min\{j_1,j_2\}}$, or we can use 6.5(0). $\square_{7.7}$

7.8 Claim. *[Long transitivity]*
Assume $1 \le \xi \le \lambda^+$. *We have* $\mathrm{WNF}^{\xi}_{\mathfrak{s}}(M_0, N_0, M_{\alpha(*)}, N_{\alpha(*)})$ *<u>when</u>:*

- (a) $\langle M_\alpha : \alpha \le \alpha(*)\rangle$ *is* $\le_{\mathfrak{s}}$*-increasing continuous*
- (b) $\langle N_\alpha : \alpha \le \alpha(*)\rangle$ *is* $\le_{\mathfrak{s}}$*-increasing continuous*
- (c) $\mathrm{WNF}^{\xi}_{\mathfrak{s}}(M_\alpha, N_\alpha, M_{\alpha+1}, N_{\alpha+1})$ *for every* $\alpha < \alpha(*)$.

Remark. 1) Recall that we do not know symmetry for $\mathrm{WNF}_{\mathfrak{s}}$ and while this claim is easy its dual is not clear at this point.

Proof. By chasing arrows. $\square_{7.8}$

7.9 Claim. *[weak existence] Assume* $1 \le \xi \le \lambda^+$.
If $(M_0, M_1, a) \le_{\mathrm{bs}} (N_0, N_1, a)$ *and* $(M_0, M_1, a) \in K^{3,\mathrm{up}}_{\mathfrak{s},\xi}$ *<u>then</u>* $\mathrm{WNF}^{\xi}_{\mathfrak{s}}(M_0, N_0, M_1, N_1)$.

Proof. Let $\alpha \le \xi$ be such that $\alpha < \lambda^+$ and $\mathbf{d}$ be a $\mathfrak{u}$-free $(0,\alpha)$-rectangle and let f be a $\le_{\mathfrak{s}}$-embedding of N_0 into $M^{\mathbf{d}}_{0,\alpha}$ such that $f(M_0) = M^{\mathbf{d}}_{0,0}$. Let $N'_1 \in K_{\mathfrak{s}}$ be $\le_{\mathfrak{s}}$-universal over $M^{\mathbf{d}}_{0,\alpha}$, exist by 5.8, and let $N'_0 = M^{\mathbf{d}}_{0,\alpha}$.

As $(N_0, N_1, a) \in K^{3,\mathrm{bs}}_{\mathfrak{s}}$ we can find a $\le_{\mathfrak{s}}$-embedding g of N_1 into N'_1 extending f such that $(g(N_0), g(N_1), g(a)) \le_{\mathrm{bs}} (N'_0, N'_1, g(a))$ so as $\le_{\mathrm{bs}}$ is a partial order preserved by isomorphisms, clearly $(g(M_0), g(M_1), g(a)) \le_{\mathrm{bs}} (N'_0, N'_1, g(a))$. Now as $(M_0, M_1, a) \in K^{3,\mathrm{up}}_{\mathfrak{s},\xi}$ it follows that $(g(M_0), g(M_1), g(a)) \in K^{3,\mathrm{up}}_{\mathfrak{s},\xi}$. Applying the definition of $K^{3,\mathrm{up}}_{\mathfrak{s},\xi}$, see Definition 6.4(1) with $g(M_0), g(M_1), g(a), N'_0, N'_1$, dual$(\mathbf{d})$

here standing for $M, N, a, M', N', \mathbf{d}$ there, recalling that $\mathfrak{u}$ is self-dual we can find a $\mathfrak{u}$-free $(\alpha+1,1)$-rectangle $\mathbf{d}^*$ and h such that: $\mathbf{d}^* \restriction (\alpha,0) = \text{dual}(\mathbf{d}), h(g(M_1) \leq_{\mathfrak{s}} M^{\mathbf{d}^*}_{0,1}, h(g(a)) = a^{\mathbf{d}^*}_{0,0}$ and h is a $\leq_{\mathfrak{s}}$-embedding of N'_1 into $M^{\mathbf{d}^*}_{\alpha+1,1}$ over $N'_0 = M^{\mathbf{d}}_{0,\alpha}$ hence $h \restriction g(M_0) = \text{ id}_{g(M_0)}$.

Now $N'_1, \mathbf{d}^+ := \text{dual}(\mathbf{d}^*) \restriction (1,\alpha)$ and $h \circ g$ are as required in Definition 7.3 (standing for $N^*, \mathbf{d}^+, g$ there) recalling 7.5. $\square_{7.9}$

7.10 Lemma. *[Amalgamation existence] Let $\xi < \lambda^+$ or just $\xi \leq \lambda^+$ and $\xi \geq 1$.*
1) If $M_0 \leq_{\mathfrak{s}} M_\ell$ for $\ell = 1,2$ and $M_1 \cap M_2 = M_0$ then for some M_3 we have $\text{WNF}^\xi_{\mathfrak{s}}(M_0, M_1, M_2, M_3)$.
2) If $M_0 \leq_{\mathfrak{s}} M_2$ then we can find an $\mathfrak{u}$-free rectangle $\mathbf{d}$ satisfying $\beta_{\mathbf{d}} = 0$ such that

$\boxtimes$ (a) $M^{\mathbf{d}}_{0,0} = M_0$
(b) $M_2 \leq_{\mathfrak{s}} M^{\mathbf{d}}_{\alpha(\mathbf{d}),0}$
(c) $(M^{\mathbf{d}}_{\alpha,0}, M^{\mathbf{d}}_{\alpha+1,0}, b^{\mathbf{d}}_{\alpha,0})$ *belongs to* $K^{3,\text{up}}_{\mathfrak{s},\xi}$ *for* $\alpha < \alpha_{\mathbf{d}}$
(d) *if M_2 is $(\lambda,*)$-brimmed over M_0 then $M^{\mathbf{d}}_2 = M^{\mathbf{d}}_{\alpha_{\mathbf{d}},0}$.*

2A) If $(M_0, M_2, b) \in K^{3,\text{bs}}_{\mathfrak{s}}$ we can add $b^{\mathbf{d}}_{0,0} = b$.
2B) Assume $K^{3,*}_{\mathfrak{s}} \subseteq K^{3,\text{bs}}_{\mathfrak{s}}$ is such that $\mathfrak{s}$ has existence for $K^{3,*}_{\mathfrak{s}}$ then in parts 2), 2A) we can replace $K^{3,\text{up}}_{\mathfrak{s},\xi}$ by $K^{3,*}_{\mathfrak{s}}$.
3) In part (1) if $(M_0, M_\ell, b_\ell) \in K^{3,\text{bs}}_{\mathfrak{s}}$ for $\ell = 1,2$ then we can add $(M_0, M_\ell, b_\ell) \leq_{\text{bs}} (M_{3-\ell}, M_3, b_\ell)$ for $\ell = 1,2$.

Proof. 1) Follows by part (3).
2) By Ax(D)(c), density, of almost good λ-frames there is $b \in M_2 \backslash M_0$ such that $\mathbf{tp}_{\mathfrak{s}}(b, M_0, M_2) \in \mathscr{S}^{\text{bs}}_{\mathfrak{s}}(M_0)$, hence $(M_0, M_2, b) \in K^{3,\text{bs}}_{\mathfrak{s}}$, by the definition of $K^{3,\text{bs}}_{\mathfrak{s}}$ it follows that so we can apply part (2A).
2A) By part (2B).
2B) So let $(M_0, M_2, b) \in K^{3,*}_{\mathfrak{s}}$ be given. We try to choose $(M_{0,\alpha}, M_{2,\alpha})$ and if $\alpha = \beta+1$ also a_β by induction on $\alpha < \lambda^+$ such that:

$\circledast$ (a) $M_{\ell,\alpha} \in K_{\mathfrak{s}}$ is $\leq_{\mathfrak{s}}$-increasing continuous for $\ell = 0,2$
(b) $M_{0,\alpha} \leq_{\mathfrak{s}} M_{2,\alpha}$
(c) $(M_{0,\alpha}, M_{2,\alpha}) = (M_0, M_2)$ for $\alpha = 0$
(d) if $\alpha = \beta+1$ then $(M_{0,\beta}, M_{0,\alpha}, a_\beta) \in K^{3,*}_{\mathfrak{s}}$ and $a_\beta \in M_{2,\alpha} \backslash M_{0,\alpha}$
(e) if $\alpha = \beta+1$ then $M_{2,\alpha}$ is $\leq_{\mathfrak{s}}$-brimmed over $M_{2,\beta}$
(f) if $\alpha = 0$ then $a_\alpha = b$.

By Fodor lemma we cannot choose for every $\alpha < \lambda^+$. For $\alpha = 0$ and α limit there are no problems, hence for some $\alpha = \beta + 1$, we have defined up to β but cannot define for α clearly $\beta < \lambda^+$. First assume

$(*)$ $M_{0,\beta} \neq M_{2,\beta}$.

So by Ax(D)(c) of Definition 5.2 of almost good λ-frame we can choose $a_\beta \in M_{2,\beta} \backslash M_{0,\beta}$ such that $\mathbf{tp}_{\mathfrak{s}}(b, M_{0,\beta}, M_{2,\beta}) \in \mathscr{S}^{\rm bs}_{\mathfrak{s}}(M_{0,\beta})$ and $a_\beta = b$ if $\beta = 0$. By the assumption on $K^{3,*}_{\mathfrak{s}}$ there is $N_\beta \in K_{\mathfrak{s}}$ such that $(M_{0,\alpha}, N_\beta, a_\beta) \in K^{3,*}_{\mathfrak{s}}$ and $\mathbf{tp}_{\mathfrak{s}}(a_\beta, M_{0,\beta}, N_\beta) = \mathbf{tp}_{\mathfrak{s}}(a_\beta, M_{0,\beta}, M_{2,\beta})$.

By the definition of orbital type (and amalgamation of $\mathfrak{K}_{\mathfrak{s}}$) without loss of generality for some $M'_{2,\beta}$ we have $N_\beta \leq_{\mathfrak{s}} M'_{2,\beta}$ and $M_{2,\beta} \leq_{\mathfrak{s}} M'_{2,\beta}$.

Let $M_{2,\alpha} \in K_{\mathfrak{s}}$ be brimmed over $M'_{2,\beta}$. So we can choose for α, contradiction.

Hence $(*)$ cannot hold so $M_{0,\beta} = M_{2,\beta}$, easily $\beta \geq 1$ (as $M_2 \neq M_0$) and by clause (e) of $\circledast$ $M_{2,\beta}$ is brimmed over $M_{2,0} = M_2$ hence over M_0. What about clause (d) of the conclusion? It follows because any two brimmed extensions of M_0 are isomorphic over it by 5.19(4) and with a little more work even over $M_0 \cup \{b\}$.

3) So let $K^{3,*}_{\mathfrak{s}} = K^{3,\rm up}_{\mathfrak{s}}$ or just $K^{3,*}_{\mathfrak{s}} \subseteq K^{3,\rm up}_{\mathfrak{s},\xi}$ and $\mathfrak{s}$ has existence for $K^{3,*}_{\mathfrak{s}}$.

Let $\mathbf{d}$ be as guaranteed in parts (2),(2A) so $a^{\mathbf{d}}_{0,0} = b_2$ and $M_0 = M^{\mathbf{d}}_{0,0}, M_2 \leq_{\mathfrak{s}} M^{\mathbf{d}}_{\alpha_{\mathbf{d}},0}$. Without loss of generality $M^{\mathbf{d}}_{\alpha_{\mathbf{d}},0} \cap M_1 = M_0$ and now we choose N_α by induction on $\alpha \leq \alpha_{\mathbf{d}}$ such that

$\boxplus$ (a) $N_\alpha \in K_{\mathfrak{s}}$ is $\leq_{\mathfrak{s}}$-increasing continuous
- (b) $M^{\mathbf{d}}_{\alpha_{\mathbf{d}},0} \cap N_\alpha = M^{\mathbf{d}}_{\alpha,0}$
- (c) $M^{\mathbf{d}}_{\alpha,0} \leq_{\mathfrak{s}} N_\alpha$
- (d) $(M^{\mathbf{d}}_{\alpha,0}, M^{\mathbf{d}}_{\alpha+1,0}, a^{\mathbf{d}}_{\alpha,0}) \leq_{\rm bs} (N_\alpha, N_{\alpha+1}, a^{\mathbf{d}}_{\alpha,0})$
- (e) $N_\alpha = M_1$ for $\alpha = 0$.

There is no problem to carry the choice by Hypothesis 7.1(4) and Definition 5.5.

Now for each $\alpha < \alpha_{\mathbf{d}}$ by clause (c) of $\boxtimes$ of part (2) or (2B) we have

$$(M^{\mathbf{d}}_{0,\alpha}, M^{\mathbf{d}}_{0,\alpha+1}, a^{\mathbf{d}}_{\alpha,0}) \in K^{3,*}_{\mathfrak{s}} \subseteq K^{3,\rm up}_{\mathfrak{s},\xi}$$

recalling the choice of $\mathbf{d}$ and by clause (d) of $\boxplus$ we have

$$(M^{\mathbf{d}}_{0,\alpha}, M^{\mathbf{d}}_{0,\alpha+1}, a^{\mathbf{d}}_{\alpha,0}) \leq_{\rm bs} (N_\alpha, N_{\alpha+1}, a^{\mathbf{d}}_{\alpha,0}),$$

hence by the weak existence Claim 7.9 we have $\mathrm{WNF}^{\xi}_{\mathfrak{s}}(M^{\mathbf{d}}_{0,\alpha}, N_\alpha, M^{\mathbf{d}}_{0,\alpha+1}, N_{\alpha+1})$.

As $\langle M^{\mathbf{d}}_{0,\alpha} : \alpha \leq \alpha_{\mathbf{d}}\rangle$ and $\langle N_\alpha : \alpha \leq \alpha_{\mathbf{d}}\rangle$ are $\leq_{\mathfrak{s}}$-increasing continuous, it follows the long transitivity claim 7.8 that $\mathrm{WNF}^{\xi}_{\mathfrak{s}}(M^{\mathbf{d}}_{0,0}, N_0, M^{\mathbf{d}}_{\alpha_{\mathbf{d}},0}, N_{\alpha(\mathbf{d})})$ which means

that $\mathrm{WNF}^\xi_{\mathfrak{s}}(M_0, M_1, M^{\mathbf{d}}_{\alpha_{\mathbf{d}},0}, N_{\alpha(\mathbf{d})})$. Let $M_3 := N_{\alpha(\mathbf{d})}$, but now $M_0 \leq_{\mathfrak{s}} M_2 \leq_{\mathfrak{s}} M^{\mathbf{d}}_{\alpha(\mathbf{d}),0}$ hence by the monotonicity Claim 7.6 we have $\mathrm{WNF}^\xi_{\mathfrak{s}}(M_0, M_1, M_2, M_3)$.

This proves the desired conclusion of part (1), but there are more demands in part (3). One is $(M_0, M_1, b_1) \leq_{\mathrm{bs}} (M_2, M_3, b_1)$, but $M_1 = N_0$ and $M_3 = N_{\alpha(\mathbf{d})}$ so this means $(M_0, N_0, b_1) \leq_{\mathrm{bs}} (M_2, N_{\alpha(\mathbf{d})}, b_1)$ and by monotonicity of non-forking it suffices to show $(M_0, N_0, b_1) \leq_{\mathrm{bs}} (M^{\mathbf{d}}_{\alpha(\mathbf{d}),0}, N_{\alpha(\mathbf{d})})$. But recall $\mathrm{WNF}^\xi_{\mathfrak{s}}(M_0, N_0, M^{\mathbf{d}}_{\alpha(\mathbf{d}),0}, N_{\alpha(\mathbf{d})})$ and this implies $(M_0, N_0, b_1) \leq_{\mathrm{bs}} (M^{\mathbf{d}}_{\alpha(\mathbf{d}),0}, N_{\alpha(\mathbf{d})}, b_1)$ by Observation 7.5(1) which as said above suffices.

Now also we have chosen $a^{\mathbf{d}}_{0,0}$ as b_2, so by clause (d) of $\boxplus$ for $\alpha = 0$ we have easily $(M_0, M^{\mathbf{d}}_{1,0}, b_2) = (M^{\mathbf{d}}_{0,0}, M^{\mathbf{d}}_{1,0}, a^{\mathbf{d}}_{0,0}) \leq_{\mathrm{bs}} (N_0, N_1, a^{\mathbf{d}}_{0,0}) = (M_1, N_1, b_2) \leq_{\mathrm{bs}} (M_1, N_{\alpha(\mathbf{d})}, b_2) = (M_1, M_3, b_2)$ but $M_0 \leq_{\mathfrak{s}} M_1 \leq_{\mathfrak{s}} M^{\mathbf{d}}_{\alpha(\mathbf{d}),0}$ so by easy monotonicity we have $(M_0, M_2, b_2) \leq_{\mathrm{bs}} (M_1, M_3, b_2)$, as desired in part (3); so we are done. $\square_{7.10}$

7.11 Remark. In the proof of 7.10(2B), if $\mathfrak{s}$ is a good λ-frame, in fact, λ steps in the induction suffice by a careful choice of a_β using bookkeeping as in the proof of 5.19(1), so we get $\alpha_{\mathbf{d}} = \lambda$. Without this extra hypothesis on $\mathfrak{s}$, this is not clear.

7.12 Claim. *Assume that $1 \leq \xi \leq \lambda^+, \alpha < \lambda^+, \mathbf{d}$ is a $\mathfrak{u}$-free $(0,\alpha)$-rectangle and $M_0 \leq_{\mathfrak{s}} M^{\mathbf{d}}_{0,0} \leq_{\mathfrak{s}} M^{\mathbf{d}}_{0,\alpha} \leq_{\mathfrak{s}} N_1$. Then we can find $\alpha', \mathbf{d}', h$ such that*

$\oplus$ (a) $\alpha' \in [\alpha, \lambda^+)$
- (b) $\mathbf{d}'$ *is a $\mathfrak{u}$-free $(0,\alpha')$-rectangle*
- (c) *h is an increasing function, $h : \alpha + 1 \to \alpha' + 1$*
- (d) $M^{\mathbf{d}'}_{0,0} = M_0$
- (e) $N_1 \leq_{\mathfrak{s}} M^{\mathbf{d}'}_{0,\alpha'}$
- (f) $i \leq \alpha \Rightarrow M^{\mathbf{d}}_{0,i} \leq_{\mathfrak{s}} M^{\mathbf{d}'}_{0,h(i)}$
- (g) *for $i < \alpha, a^{\mathbf{d}}_{0,i} = a^{\mathbf{d}'}_{0,h(i)}$ and $\mathbf{tp}_{\mathfrak{s}}(a^{\mathbf{d}'}_{0,h(i)}, M^{\mathbf{d}'}_{0,h(i)}, M^{\mathbf{d}'}_{0,h(i)+1})$ does not fork over $M^{\mathbf{d}}_{0,i}$*
- (h) $(M^{\mathbf{d}'}_{0,\beta}, M^{\mathbf{d}'}_{0,\beta+1}, a^{\mathbf{d}'}_{0,\beta}) \in K^{3,\mathrm{up}}_{\mathfrak{s},\xi}$ *for $\beta < \alpha'$.*

Proof. We can choose M'_i by induction on $i \leq 1 + \alpha + 1$ such that

(α) $\langle M'_j : j \leq i \rangle$ is increasing continuous

(β) $M'_0 = M_0$

(γ) M'_i is brimmed over M'_j if $i = j+1 \le 1+\alpha+1$

(δ) $M_\alpha \cap M'_{1+i} = M^{\mathbf{d}}_{0,i}$ if $i \le \alpha$

(ε) $M^{\mathbf{d}}_{0,i} \le_{\mathfrak{s}} M'_{1+i}$ if $i \le \alpha$

(ζ) $\mathbf{tp}_{\mathfrak{s}}(a^{\mathbf{d}}_{0,i}, M'_i, M'_{i+1})$ does not fork over $M^{\mathbf{d}}_{0,i}$ if $i < \alpha$

(η) $N_1 \le_{\mathfrak{s}} M'_{1+\alpha+1}$.

This is possible because we have disjoint amalgamation (see 5.23). Now for each $i \le 1+\alpha$ use 7.10(2A) with $M'_i, M'_{i+1}, a^{\mathbf{d}}_{0,i}$ here standing for M_0, M_2, b there (so clause (d) there apply). $\square_{7.12}$

7.13 Remark. Recall that from now on we are assuming that $\mathfrak{K}_{\mathfrak{s}}$ is categorical.

7.14 Claim. *[Symmetry] There is $\xi = \xi_{\mathfrak{s}} < \lambda^+$ such that ($\xi \ge \xi^{\rm rd}_{\mathfrak{s}}$ for simplicity, see 5.21 and) for every $\zeta < \lambda^+$, if $\mathrm{WNF}^{\xi}_{\mathfrak{s}}(M_0, N_0, M_1, N_1)$ <u>then</u> $\mathrm{WNF}^{\zeta}_{\mathfrak{s}}(M_0, M_1, N_0, N_1)$ holds.*

Remark. 1) Yes, the models N_0, M_1 exchange places.
2) Without categoricity, $\xi = \xi_{\mathfrak{s}, M_0}$ is O.K.

Proof. By 7.10(2) there are $\xi = \xi(*) < \lambda^+$ and $\mathbf{d}$, a $\mathfrak{u}$-free $(0,\xi)$-rectangle with each $(M^{\mathbf{d}}_{0,\alpha}, M^{\mathbf{d}}_{0,\alpha+1}, a^{\mathbf{d}}_{0,\alpha})$ belonging to $K^{3,\rm up}_{\mathfrak{s},\zeta}$ for every $\zeta < \lambda^+$ such that $M^{\mathbf{d}}_{0,\xi}$ is brimmed over $M^{\mathbf{d}}_{0,0}$ and $M_0 = M^{\mathbf{d}}_{0,0}$. Note that the choice of ξ does not depend on $\zeta, \langle M_0, N_0, M_1, N_1\rangle$, just on M_0 by 7.10 and it does not depend on M_0 recalling $K_{\mathfrak{s}}$ is categorical.

As $M^{\mathbf{d}}_{0,\xi}$ is $\le_{\mathfrak{s}}$-universal over $M^{\mathbf{d}}_{0,0}$ without loss of generality $N_0 \le_{\mathfrak{s}} M^{\mathbf{d}}_{0,\xi}$.

Now let $\zeta < \lambda^+$ and recall that we assume $\mathrm{WNF}^{\xi}_{\mathfrak{s}}(M_0, N_0, M_1, N_1)$. Let $\mathbf{d}^+, N^+, f$ be as guaranteed by Definition 7.3(1) and by renaming without loss of generality the function f is the identity. Now for each $\alpha < \xi$, we shall apply the weak existence claim 7.9, with $M^{\mathbf{d}^+}_{0,\alpha}, M^{\mathbf{d}^+}_{1,\alpha}, M^{\mathbf{d}^+}_{0,\alpha+1}, M^{\mathbf{d}^+}_{1,\alpha+1}, a^{\mathbf{d}^+}_{0,\alpha}$ here standing for M_0, N_0, M_1, N_1, a as there; this is O.K. as its assumptions mean $(M^{\mathbf{d}^+}_{0,\alpha}, M^{\mathbf{d}^+}_{0,\alpha+1}, a^{\mathbf{d}^+}_{0,\alpha}) \le_{\rm bs} (M^{\mathbf{d}^+}_{1,\alpha}, M^{\mathbf{d}^+}_{1,\alpha+1}, a^{\mathbf{d}^+}_{0,\alpha})$ and $(M^{\mathbf{d}^+}_{0,\alpha}, M^{\mathbf{d}^+}_{0,\alpha+1}, a^{\mathbf{d}^+}_{0,\alpha}) \in K^{3,\rm up}_{\mathfrak{s},\zeta}$ which hold by clause $\boxtimes(c)$ of Claim 7.10(2), i.e. by the choice of $\mathbf{d}$ as $\mathbf{d}^+ \restriction (0, \alpha(\mathbf{d})) = \mathbf{d}$. Hence the conclusion of 7.9 applies, which gives that we have $\mathrm{WNF}^{\zeta}_{\mathfrak{s}}(M^{\mathbf{d}^+}_{0,\alpha}, M^{\mathbf{d}^+}_{1,\alpha}, M^{\mathbf{d}^+}_{0,\alpha+1}, M^{\mathbf{d}^+}_{1,\alpha+1})$. Of course $\langle M^{\mathbf{d}^+}_{0,\alpha} : \alpha \le \xi\rangle$ and $\langle M^{\mathbf{d}^+}_{1,\alpha} : \alpha \le \xi\rangle$ are $\le_{\mathfrak{s}}$-increasing continuous. Together by the long transitivity, claim 7.8 we have $\mathrm{WNF}^{\zeta}_{\mathfrak{s}}(M^{\mathbf{d}^+}_{0,0}, M^{\mathbf{d}^+}_{1,0}, M^{\mathbf{d}^+}_{0,\xi}, M^{\mathbf{d}^+}_{1,\xi})$. But

$M^{\mathbf{d}^+}_{0,0} = M_0, M^{\mathbf{d}^+}_{1,0} = M_1$ and $N_0 \le_{\mathfrak{s}} M^{\mathbf{d}}_{0,\xi}$ and $N_1 \le_{\mathfrak{s}} N^+, M^{\mathbf{d}^+}_{1,\xi} \le_{\mathfrak{s}} N^+$ so by the monotonicity claim, 7.6, we have $\mathrm{WNF}^{\zeta}_{\mathfrak{s}}(M_0, M_1, N_0, N_1)$ as required. $\square_{7.14}$

7.15 Conclusion. If $\mathrm{WNF}^{\xi}_{\mathfrak{s}}(M_0, N_0, M_1, N_1)$ and $\xi \ge \xi_{\mathfrak{s}}$, see 7.14 then $\zeta < \lambda^+ \Rightarrow \mathrm{WNF}^{\zeta}_{\mathfrak{s}}(M_0, N_0, M_1, N_1)$ that is $\mathrm{WNF}_{\mathfrak{s}}(M_0, N_0, M_1, N_1)$.

Proof. Applying 7.14 twice recalling 7.5(3) in the end. $\square_{7.15}$

7.16 Claim. *(*$\mathrm{WNF}_{\mathfrak{s}}$ *lifting or weak uniqueness)*

If $\mathrm{WNF}_{\mathfrak{s}}(M_0, N_0, M_1, N_1)$ *and* $\alpha < \lambda^+$ *and* $\langle M_{0,i} : i \le \alpha \rangle$ *is* $\le_{\mathfrak{s}}$*-increasing continuous,* $M_{0,0} = M_0$ *and* $M_{0,\alpha} = N_0$ *then we can find a* $\le_{\mathfrak{s}}$*-increasing continuous sequence* $\langle M_{1,i} : i \le \alpha + 1 \rangle$ *such that* $M_{1,0} = M_1, N_1 \le_{\mathfrak{s}} M_{1,\alpha+1}$ *and for each* $i < \alpha$ *we have* $\mathrm{WNF}_{\mathfrak{s}}(M_{0,i}, M_{0,i+1}, M_{1,i}, M_{1,i+1})$ *for* $i < \alpha$.

Proof. We shall use 7.12.

By induction on $i \le \alpha$ we can find M'_i which is $\le_{\mathfrak{s}}$-increasing continuous such that $M'_i \cap N_1 = M_{0,i}, M_{0,i} \le_{\mathfrak{s}} M'_i$ and if $i = j+1$ then M'_i is brimmed over M'_j and $\mathrm{WNF}_{\mathfrak{s}}(M_{0,i}, M_{0,i+1}, M'_i, M'_{i+1})$.

So by 7.10(2) and see 7.14 we can find a $\mathfrak{u}_{\mathfrak{s}}$-free $(0, \xi_{\mathfrak{s}}\alpha)$-rectangle $\mathbf{d}$ such that $M^{\mathbf{d}}_{\xi_{\mathfrak{s}} i} = M'_i$ for $i \le \xi_{\mathfrak{s}}\alpha$ and $(M^{\mathbf{d}}_{\varepsilon,0}, M^{\mathbf{d}}_{\varepsilon+1,0}, a^{\mathbf{d}}_{\varepsilon,0}) \in K^{3,\mathrm{up}}_{\mathfrak{s}}$ for $\varepsilon < \xi_{\mathfrak{s}}\alpha$. Recalling $M_{0,\alpha} = N_0$, without loss of generality $M^{\mathbf{d}}_{0,\xi_{\mathfrak{s}}\alpha} \cap N_1 = N_0$ so by 7.10(1) we can find N^*_1 such that $\mathrm{WNF}_{\mathfrak{s}}(N_0, N_1, M^{\mathbf{d}}_{0,\xi_{\mathfrak{s}}\alpha}, N^*_1)$. Recalling that we are assuming $\mathrm{WNF}_{\mathfrak{s}}(M_0, N_0, M_1, N_1)$, by symmetry, i.e. 7.14 we have $\mathrm{WNF}_{\mathfrak{s}}(M_0, M_1, N_0, N_1)$ hence by transitivity, i.e. Claim 7.8 we can deduce that $\mathrm{WNF}_{\mathfrak{s}}(M_0, M_1, M^{\mathbf{d}}_{0,\xi_{\mathfrak{s}}\alpha}, N^*_1)$ hence by 7.14, i.e. symmetry $\mathrm{WNF}_{\mathfrak{s}}(M_0, M^{\mathbf{d}}_{0,\xi_{\mathfrak{s}}\alpha}, M_1, N^*_1)$. By the definition of $\mathrm{WNF}_{\mathfrak{s}}$, we can find a $\mathfrak{u}_{\mathfrak{s}}$-free $(1, \xi_{\mathfrak{s}}\alpha + 1)$-rectangle $\mathbf{d}^+$ such that $\mathbf{d}^+ \restriction (0, \xi_{\mathfrak{s}}\alpha) = \mathbf{d}$ and $N^*_1 \le_{\mathfrak{s}} M^{\mathbf{d}^+}_{1,\xi_{\mathfrak{s}}\alpha+1}$ and $M_1 = M^{\mathbf{d}^+}_{1,0}$.

By the weak existence claim 7.9, we have $\mathrm{WNF}_{\mathfrak{s}}(M^{\mathbf{d}^+}_{0,\varepsilon}, M^{\mathbf{d}^+}_{1,\varepsilon}, M^{\mathbf{d}^+}_{0,\varepsilon+1}, M^{\mathbf{d}^+}_{1,\varepsilon+1})$ for each $\varepsilon < \xi_{\mathfrak{s}}\alpha$.

Let $M_{1,\alpha+1} = M^{\mathbf{d}^+}_{1,\xi_{\mathfrak{s}}\alpha+1}$ and for $i \le \alpha$ let $M_{1,i} := M^{\mathbf{d}^+}_{1,\xi_{\mathfrak{s}} i}$. So clearly $\langle M_{1,i} : i \le \alpha + 1 \rangle$ is $\le_{\mathfrak{s}}$-increasing continuous, $M_{1,0} = M_1, N_1 \le_{\mathfrak{s}} M_{1,\alpha+1}$. Now to finish the proof we need to show, for $i < \alpha$ that $\mathrm{WNF}_{\mathfrak{s}}(M_{0,i}, M_{0,i+1}, M_{1,i}, M_{1,i+1})$.

For each $i < \alpha$ by the long transitivity claim, i.e. 7.8 applied to $\langle M^{\mathbf{d}^+}_{0,\xi_{\mathfrak{s}} i+\varepsilon} : \varepsilon \le \xi_{\mathfrak{s}} \rangle$ and $\langle M^{\mathbf{d}^+}_{1,\xi_{\mathfrak{s}},i+\varepsilon} : \varepsilon \le \xi_{\mathfrak{s}} \rangle$ we have $\mathrm{WNF}_{\mathfrak{s}}(M^{\mathbf{d}^+}_{0,\xi_{\mathfrak{s}} i}, M^{\mathbf{d}^+}_{1,\xi_{\mathfrak{s}} i}, M^{\mathbf{d}^+}_{0,\xi_{\mathfrak{s}}(i+1)}, M^{\mathbf{d}^+}_{1,\xi_{\mathfrak{s}}(i+1)})$, by symmetry we have $\mathrm{WNF}_{\mathfrak{s}}(M^{\mathbf{d}^+}_{0,\xi_{\mathfrak{s}} i}, M^{\mathbf{d}^+}_{0,\xi_{\mathfrak{s}}(i+1)}, M^{\mathbf{d}^+}_{1,\xi_{\mathfrak{s}} i}, M^{\mathbf{d}^+}_{1,\xi_{\mathfrak{s}}(i+1)})$ which means $\mathrm{WNF}_{\mathfrak{s}}(M'_i, M'_{i+1}, M_{1,i}, M_{i,1+i})$.

Recall that for each $i < \alpha$ we have $\mathrm{WNF}_{\mathfrak{s}}(M_{0,i}, M_{0,i+1}, M'_i, M'_{i+1})$. By the transitivity claim 7.8, the two previous sentences imply $\mathrm{WNF}_{\mathfrak{s}}(M_{0,i}, M_{0,i+1}, M_{1,i}, M_{1,i+1})$ as required. $\square_{7.16}$

7.17 Theorem. *1)* $\mathrm{WNF}_{\mathfrak{s}}$ *is a weak non-forking relation on* $\mathfrak{K}_\lambda$ *respecting* $\mathfrak{s}$ *and having disjointness (see Definition 7.18 below).*
2) If NF *is a weak non-forking relation of* $\mathfrak{K}_\lambda$ *respecting* $\mathfrak{s}$*, then* $\mathrm{NF} \subseteq \mathrm{WNF}_{\mathfrak{s}}$*.*

A relative of Definition III.600-nf.0X is:

7.18 Definition. Let $\mathfrak{K}_\lambda$ be a λ-a.e.c.
1) We say that NF is a weak non-forking relation on $\mathfrak{K}_\lambda$ when: it satisfied the requirements in Definition III.600-nf.0X(1); except that we replace uniqueness (clause (g) there) by weak uniqueness meaning that the conclusion of Claim 7.16 holds (replacing $\mathrm{WNF}_{\mathfrak{s}}$ by NF); or see the proof of 7.17 below for a list or use Definition III.600-nf.20.9(1).
2) Let $\mathfrak{t}$ be an almost good λ-frame and $\mathfrak{K}_\lambda = \mathfrak{K}_{\mathfrak{t}}$ and NF be a weak non-forking relation on $\mathfrak{K}_\lambda$. We say that NF respects $\mathfrak{t}$ when: if $\mathrm{NF}(M_0, N_0, M_1, N_1)$ and $(M_0, N_0, a) \in K^{3,\mathrm{bs}}_{\mathfrak{t}}$ then $\mathbf{tp}_{\mathfrak{t}}(a, M_1, N_1)$ does not fork over M_0. We say NF is a weak $\mathfrak{t}$-non-forking relation when it is a weak $\mathfrak{t}$-non-forking relation respecting $\mathfrak{t}$.
3) In part (1) we say NF has disjointness when $\mathrm{WNF}(M_0, N_0, M_1, N_1) \Rightarrow M_0 \cap M_1 = M_0$.
4) We say NF is a pseudo non-forking relation on $\mathfrak{K}_\lambda$ when we have clauses (a)-(f) of Definition III.600-nf.0X or see the proof below. Also here parts (2),(3) are meaningful.

Proof of 7.17. 1) Let us list the conditions on $\mathrm{NF} := \mathrm{WNF}_{\mathfrak{s}}$ being a weak non-forking relation let $\mathfrak{K}_\lambda = \mathfrak{K}_{\mathfrak{s}}$. We shall use 7.15 freely.

Condition (a): NF is a 4-place relation on $\mathfrak{K}_\lambda$.
[Why? This holds by Definition 7.3(1),(2).]

Condition (b): $\mathrm{NF}(M_0, M_1, M_2, M_3)$ implies $M_0 \leq_{\mathfrak{K}_\lambda} M_\ell \leq_{\mathfrak{K}_\lambda} M_3$ for $\ell = 1, 0$ and NF is preserved by isomorphisms.
[Why? The preservation by isomorphisms holds by the definition, and also the order demands.]

Condition $(c)_1$: [Monotonicity] If $\mathrm{NF}(M_0, M_1, M_2, M_3)$ and $M_0 \leq_{\mathfrak{K}_\lambda} M'_\ell \leq_{\mathfrak{K}_\lambda} M_\ell$ for $\ell = 1, 2$ then $\mathrm{NF}(M_0, M'_1, M'_2, M_3)$.
[Why? By 7.6.]

Condition (c)$_2$: [Monotonicity] If $\mathrm{NF}(M_0, M_1, M_2, M_3)$ and $M_3 \le_{\mathfrak{K}_\lambda} M_3'$ and $M_1 \cup M_2 \subseteq M_3'' \le_{\mathfrak{K}_\lambda} M_3$ then $\mathrm{NF}(M_0, M_1, M_2, M_3'')$.
[Why? By Claim 7.6.]

Condition (d): [Symmetry] If $\mathrm{NF}(M_0, M_1, M_2, M_3)$ then $\mathrm{NF}(M_0, M_2, M_1, M_3)$.
[Why? By Claim 7.14.]

Condition (e): [Long Transitivity] If $\alpha < \lambda^+$, $\mathrm{NF}(M_i, N_i, M_{i+1}, N_{i+1})$ for each $i < \alpha$ and $\langle M_i : i \le \alpha\rangle, \langle N_i : i \le \lambda\rangle$ are $\le_{\mathfrak{K}_\lambda}$-increasing continuous sequences then $\mathrm{NF}(M_0, N_0, M_\alpha, N_\alpha)$.
[Why? By Claim 7.8.]

Condition (f): [Existence] Assume $M_0 \le_{\mathfrak{K}_\lambda} M_\ell$ for $\ell = 1, 2$. Then for some M_3, f_1, f_2 we have $M_0 \le_{\mathfrak{K}_\lambda} M_3 \in \mathfrak{K}_\lambda, f_\ell$ is a $\le_{\mathfrak{K}_\lambda}$-embedding for M_ℓ into M_3 over M_0 for $\ell = 1, 2$ and $\mathrm{NF}(M_0, f_1(M_1), f_2(M_2), M_3)$. Here we have the disjoint version, i.e. $f_2(M_1) \cap f_2(M_2) = M_2$.
[Why? By Lemma 7.10(1).]

Condition (g): Lifting or weak uniqueness [a replacement for uniqueness]

This is the content of 7.16.

Thus we have finished presenting the definition of "NF is a weak non-forking relation on $\mathfrak{K}_{\mathfrak{s}}$" and proving that $\mathrm{WNF}_{\mathfrak{s}}$ satisfies those demands.

But we still owe "$\mathrm{WNF}_{\mathfrak{s}}$ respect $\mathfrak{s}$" where NF respect $\mathfrak{s}$ means that if $\mathrm{NF}(M_0, N_0, M_1, N_1)$ and $(M_0, N_0, a) \in K^{3,\mathfrak{s}}_{\mathfrak{s}}$ then $\mathbf{tp}_{\mathfrak{s}}(a, M_1, N_1)$ does not fork over M_0, i.e. $(M_0, N_0, a) \le_{\mathrm{bs}} (M_1, N_1, a) \in K^{3,\mathrm{bs}}_{\mathfrak{s}}$.
[Why? This holds by Observation 7.5(1).]

Also the disjointness of WNF is easy; use 7.7 and categoricity.

Proof of 7.17(2).

So assume $\mathrm{NF}(M_0, N_0, M_1, N_1)$ and we should prove $\mathrm{WNF}_{\mathfrak{s}}(M_0, N_0, M_1, N_1)$ so let $\mathbf{d}$ be as in Definition 7.3(1). As NF satisfies existence, transitivity and monotonicity without loss of generality it suffices to deal with the case $M^{\mathbf{d}}_{0,\alpha(\mathbf{d})} = N_0$.

This holds by the definition of $\mathrm{WNF}_{\mathfrak{s}}$ in 7.3 and clause (g) in the definition of being weak non-forking relation and "respecting $\mathfrak{s}$". $\square_{7.17}$

At last we can get rid of the "almost" in "almost good λ-frame", of course, this is under the Hypothesis 7.1, otherwise we do not know.

7.19 Lemma. *1) If* $\mathfrak{t}$ *is an almost good* λ*-frame and* WNF *is a weak non-forking relation on* $\mathfrak{K}_\lambda$ *respecting* $\mathfrak{t}$ *then* $\mathfrak{t}$ *is a good* λ*-frame.*
2) In 7.1, $\mathfrak{s}$ *is a good* λ*-frame.*

Proof. Part (2) follows from part (1) and 7.17(1) above. So recalling Definition 5.2 we should just prove that $\mathfrak{t}$ satisfies Ax(E)(c). Note that the Hypothesis 7.20 below holds hence we are allowed to use 7.25.

Let $\langle M_i : i \le \delta \rangle$ be $\le_{\mathfrak{s}}$-increasing continuous and $p \in \mathscr{S}^{\rm bs}_{\mathfrak{s}}(M_\delta)$ and we should prove that p does not fork over M_i for some $i < \delta$. By renaming without loss of generality $\delta < \lambda^+$ and δ is divisible by $\lambda^2\omega$ and $\varepsilon \le \lambda \wedge i < \delta \Rightarrow M_{i+1} = M_{i+1+\varepsilon}$. Let $\mathfrak{u}$ be as in Definition 7.2, so $\mathfrak{u}$ is a nice construction framework. Let $\alpha = \delta, \beta = \delta$.

Now

$(*)_1$ there is $\mathbf{d}$ such that

- (α) $\mathbf{d}$ is a $\mathfrak{u}$-free (α,β)-rectangle
- (β) $\mathbf{d}$ is a strictly brimmed, see Definition 5.15(2)
- (γ) if $i < \alpha$ and $j < \delta$ then $\mathrm{WNF}(M^{\mathbf{d}}_{i,j}, M^{\mathbf{d}}_{i,j+1}, M^{\mathbf{d}}_{i+1,j}, M^{\mathbf{d}}_{i+1,j+1})$
- (δ) $M^{\mathbf{d}}_{i,0} = M_i$ for $i \le \delta$
- (ε) $\mathbf{J}^{\mathbf{d}}_{i,j} = \emptyset$ when $i < \delta, j \le \delta$ and $\mathbf{I}^{\mathbf{d}}_{i,j} = \emptyset$ when $i \le \delta, j < \delta$.

This will be done as in the proof of Observation 5.17.

By the properties of WNF (i.e. using twice long transitivity and symmetry)

$(*)_2$ if $i_1 < i_2 \le \alpha$ and $j_1 < j_2 \le \beta$ <u>then</u> $\mathrm{WNF}(M^{\mathbf{d}}_{i_1,j_1}, M^{\mathbf{d}}_{i_1,j_2}, M^{\mathbf{d}}_{i_2,j_1}, M^{\mathbf{d}}_{i_2,j_2})$ and $\mathrm{WNF}(M^{\mathbf{d}}_{i_1,j_1}, M^{\mathbf{d}}_{i_2,j_1}, M^{\mathbf{d}}_{i_1,j_2}, M^{\mathbf{d}}_{i_2,j_2})$.

Now we can choose a $\mathfrak{u}$-free $(\alpha_{\mathbf{d}}, \beta_{\mathbf{d}})$-rectangle $\mathbf{e}$ such that

$(*)_3$ (a) $M^{\mathbf{e}}_{i,j} = M^{\mathbf{d}}_{i,j}$ for $i \le \alpha_{\mathbf{d}}, j \le \beta_{\mathbf{d}}$

(b) if $i < \alpha, j < \beta$ and $p \in \mathscr{S}^{\rm bs}_{\mathfrak{s}}(M^{\mathbf{d}}_{i,j})$ then for λ ordinals $\varepsilon < \lambda$ we have:

- (α) $\mathbf{tp}_{\mathfrak{t}}(b^{\mathbf{d}}_{i+\varepsilon,j}, M^{\mathbf{d}}_{i+\varepsilon,j+1}, M^{\mathbf{d}}_{i+\varepsilon+1,j+1})$ is a non-forking extension of p, recalling $\mathbf{J}^{\mathbf{e}}_{i+\varepsilon,j} = \{b^{\mathbf{e}}_{i+\varepsilon,j}\}$,
- (β) $\mathbf{tp}_{\mathfrak{t}}(a^{\mathbf{e}}_{i,j+\varepsilon}, M^{\mathbf{d}}_{i+1,j+\varepsilon}, M^{\mathbf{d}}_{i+1,j+\varepsilon+1})$ is a non-forking extension of p, recalling $\mathbf{I}^{\mathbf{e}}_{i,j+\varepsilon} = \{a^{\mathbf{e}}_{i+1,j+\varepsilon}\}$.

[Why? We can choose $\mathbf{e} \restriction (\alpha,\alpha)$ by induction on $\alpha \le \delta$. The non-forking condition in the definition of $\mathfrak{u}$-free holds because WNF respects $\mathfrak{t}$ and $(*)_2$.]

So $\mathbf{d}$ is full (see Definition 5.15(3),(3A), even strongly full) hence by Claim 5.19(3A)(c)

$(*)_4$ $M^{\mathbf{d}}_{\alpha,\beta}$ is brimmed over $M^{\mathbf{d}}_{\alpha,0} = M_\delta$.

Hence $p \in \mathscr{S}^{\rm bs}_{\mathfrak{s}}(M_\delta) = \mathscr{S}^{\rm bs}_{\mathfrak{t}}(M^{\mathbf{d}}_{\alpha,0})$ is realized in $M_{\alpha,\beta}$ say by $c \in M_{\alpha,\beta}$, so for some $i < \alpha$ we have $c \in M_{i,\beta}$. As $\mathrm{WNF}(M^{\mathbf{d}}_{i,0}, M^{\mathbf{d}}_{i,\beta}, M^{\mathbf{d}}_{\alpha,0}, M^{\mathbf{d}}_{\alpha,\beta})$ holds by $(*)_2$ above by Claim 7.25 below, it follows that $\mathbf{tp}_{\mathfrak{t}}(c, M^{\mathbf{d}}_{\alpha,0}, M^{\mathbf{d}}_{\alpha,\beta})$ does not fork over $M^{\mathbf{d}}_{i,0}$ which means $i < \delta$ and p does not fork over M_i as required. $\square_{7.19}$

Now for the rest of the section we replace Hypothesis 7.1 by

7.20 Hypothesis. Assume $\mathfrak{s}$ is an almost good λ-frame categorical in λ and WNF is a weak non-forking relation on $\mathfrak{K}_{\mathfrak{s}}$ respecting $\mathfrak{s}$.

The following is related to the proof of 7.19.

7.21 Definition. 1) We say $\mathbf{d} = \langle M_{\alpha,\beta} : \alpha \le \alpha_{\mathbf{d}}, \beta \le \beta_{\mathbf{d}}\rangle$ is a WNF-free rectangle (or $(\alpha_{\mathbf{d}}, \beta_{\mathbf{d}})$-rectangle) when:

- (a) $\langle M_{\alpha,j} : j \le \beta_{\mathbf{d}}\rangle$ is $\le_{\mathfrak{s}}$-increasing continuous for each $\alpha \le \alpha_{\mathbf{d}}$
- (b) $\langle M_{i,\beta} : i \le \alpha_{\mathbf{d}}\rangle$ is $\le_{\mathfrak{s}}$-increasing continuous for each $\beta \le \beta_{\mathbf{d}}$
- (c) $\mathrm{WNF}(M_{i,j}, M_{i+1,j}, M_{i,j+1}, M_{i+1,j+1})$ for $i < \alpha, j < \beta$.

2) Let $\bar\alpha = \langle \alpha_j : j \le \beta\rangle$ be $\le$-increasing.

We say $\mathbf{d} = \langle M_{i,j} : i \le \alpha_j$ and $j \le \beta\rangle$ is a WNF-free $(\langle \alpha_j : j \le \beta\rangle, \beta)$-triangle <u>when</u>:

- (a) $\langle M_{i,j} : i \le \alpha_j\rangle$ is $\le_{\mathfrak{s}}$-increasing continuous for each $j \le \beta$
- (b) $\langle M_{i,j} : j \le \beta$ satisfies $i \le \alpha_j\rangle$ is $\le_{\mathfrak{s}}$-increasing continuous for each $i < \alpha_\beta$
- (c) $\mathrm{WNF}(M_{i,j}, M_{i+1,j}, M_{i,j+1}, M_{i+1,j+1})$ for $j < \beta, i < \alpha_j$.

Now we may note that some facts proved in Chapter IV for weakly successful good λ-frame can be proved under Hypothesis 7.1 or just 7.20. Systematically see [Sh 842].

7.22 Claim. *$M^{\mathbf{d}}_{\alpha_\beta,\beta}$ is brimmed over $M_{0,\beta}$ <u>when</u>:*

- (a) *$\bar\alpha = \langle \alpha_j : \alpha \le \beta\rangle$ is increasing continuous*
- (b) *$\mathbf{d}$ is a WNF-free $(\bar\alpha, \beta)$-triangle*
- (c) *$M_{i+1,j+1}$ is $\le_{\mathfrak{s}}$-universal over $M_{i,j}$ when $i < \alpha_j, j < \beta$.*

Proof. If β and each α_j is divisible by λ we can repeat (part of the) proof of 7.19 and this suffices for proving 7.25 hence for proving 7.19 (no vicious circle!)

In general we have to repeat the proof of 5.19 <u>or</u> first find a WNF-free $(\langle \lambda\alpha_j : j \le \beta\rangle, \beta)$-rectangle $\mathbf{d}'$ which is brimmed and full and $M^{\mathbf{d}'}_{\lambda,j} = M^{\mathbf{d}^+}_{i,j}$ for $i \le \alpha_j, j \le \beta$ and then use the first sentence. $\square_{7.22}$

It is natural to replace $\le_{\rm bs}$ by the stronger $\le_{\rm wnf}$ defined below (and used later).

7.23 Definition. 1) Let $\le_{\rm wnf}$ be the following two-place relation on $K^{2,{\rm bs}}_{\mathfrak{s}} := \{(M,N) : M \le_{\mathfrak{s}} N$ are from $\mathfrak{K}_{\mathfrak{s}}\}$, we have $(M_0,N_0) \le_{\rm wnf} (M_1,N_1)$ iff:

- (a) $(M_\ell,N_\ell) \in K^{2,{\rm bs}}_{\mathfrak{s}}$ for $\ell = 0,1$
- (b) $\mathrm{WNF}(M_0,N_0,M_1,N_1)$.

2) Let $(M_1,N_1,a) \le_{\rm wnf} (M_2,N_2,a)$ means $(M_\ell,N_\ell,a) \in K^{3,{\rm bs}}_{\mathfrak{s}}$ for $\ell = 1,2$ and $(M_1,N_1) \le_{\rm wnf} (M_2,N_2)$.

7.24 Claim. *1) $\le_{\rm wnf}$ is a partial order on $K^{2,{\rm bs}}_{\mathfrak{s}}$.*
2) If $\langle (M_\alpha,N_\alpha) : \alpha < \delta\rangle$ is $\le_{\rm wnf}$-increasing continuous <u>then</u> $\alpha < \delta \Rightarrow (M_\alpha,N_\alpha) \le_{\rm wnf} (\bigcup\limits_{\beta<\delta} M_\beta, \bigcup\limits_{\beta<\delta} N_\beta) \in K^{3,{\rm bs}}_{\mathfrak{s}}$.
3) If $(M_1,N_1) \le_{\rm wnf} (M_2,N_2)$ and $(M_1,N_1,a) \in \mathfrak{K}^{3,{\rm bs}}_{\mathfrak{s}}$ <u>then</u> $(M_1,N_1,a) \le_{\rm bs} (M_2,N_2,a) \in K^{3,{\rm bs}}_{\mathfrak{s}}$.
4) If $M \le_{\mathfrak{s}} N' \le_{\mathfrak{s}} N$ and $(M,N) \in K^{2,{\rm bs}}_{\mathfrak{s}}$ <u>then</u> $(M,N') \le_{\rm wnf} (M,N)$.
5) If $(M_1,N_1) \le_{\rm wnf} (M_2,N_2)$ and $M_1 \le_{\mathfrak{s}} N'_1 \le_{\mathfrak{s}} N_1, M_1 \le_{\mathfrak{s}} M'_2 \le_{\mathfrak{s}} M_2$ and $N'_1 \cup M'_2 \subseteq N'_2 \le_{\mathfrak{s}} N''_2$ and $N_2 \le_{\mathfrak{s}} N''_2$ <u>then</u> $(M_1,N'_1) \le_{\rm wnf} (M'_2,N'_2)$.
6) Similarly to (1),(2),(4),(5) for $(K^{3,{\rm bs}}_{\mathfrak{s}}, \le_{\rm wnf})$.

Proof. Easy by now. $\square_{7.24}$

The following is a "downward" version of "WNF respect $\mathfrak{s}$" which was used in the proof of 7.19.

7.25 Claim. *If $\mathrm{WNF}(M_0,N_0,M_1,N_1)$ and $c \in N_0$ and $(M_1,N_1,a) \in K^{3,{\rm bs}}_{\mathfrak{s}}$ <u>then</u> $(M_0,N_0,c) \in K^{3,{\rm bs}}_{\mathfrak{s}}$ and $\mathbf{tp}_{\mathfrak{s}}(c,M_1,N_1)$ does not fork over M_0.*

Proof. The second phrase in the conclusion (about "$\mathbf{tp}_{\mathfrak{s}}(a,M_1,N_1)$ does not fork over M_0") follows from the first by "WNF respects $\mathfrak{t}$"; so if $\mathfrak{s}$ is type full (i.e. $\mathscr{S}^{\rm bs}_{\mathfrak{s}}(M) = \mathscr{S}^{\rm na}_{\mathfrak{s}}(M)$) this is easy. (So if $\mathfrak{s}$ is type-full the result is easy.)

Toward contradiction assume

$(*)_0$ (M_0,N_0,M_1,N_1,c) form a counterexample.

Also by monotonicity properties without loss of generality :

$(*)_1$ (a) N_0 is brimmed over M_0
(b) M_1 is brimmed over M_0
(c) N_1 is brimmed over $N_0 \cup M_1$.

Let $\langle \mathscr{U}_\varepsilon : \varepsilon < \lambda_{\mathfrak{s}} \rangle$ be an increasing sequence of subsets of $\lambda_{\mathfrak{s}}$ such that $|\mathscr{U}_0| = |\mathscr{U}_{\varepsilon+1} \backslash \mathscr{U}_\varepsilon| = \lambda_{\mathfrak{s}}$. We now by induction on $\varepsilon \le \lambda$ choose $\mathbf{d}_\varepsilon, \bar{\mathbf{a}}_\varepsilon$ such that $\mathbf{d}_\varepsilon = \langle M_{i,j} : i \le \lambda j, j \le \varepsilon \rangle$ and:

$\circledast$ (a) $\mathbf{d}_\varepsilon$ is a WNF-free triangle
(b) $M_{i+1,j+1}$ is brimmed over $M_{i+1,j} \cup M_{i,j+1}$ when $i < \lambda j \wedge j < \varepsilon$
(c) $M_{0,j+1}$ is brimmed over $M_{0,j}$
(d) $\bar{\mathbf{a}}_\varepsilon = \langle a_\alpha : \alpha \in \mathscr{U}_\varepsilon \rangle$ list the elements of $M_{\varepsilon\lambda,\varepsilon}$
(e) if $\varepsilon = j+1$ and $\mathscr{W}_j \subseteq \mathscr{U}_j$ defined below is $\neq \emptyset$ and $\gamma_j = \min(\mathscr{W}_j)$ then $\mathbf{tp}_{\mathfrak{s}}(a_{\gamma_j}, M_{0,\varepsilon}, M_{\lambda j,\varepsilon}) \in \mathscr{S}^{\mathrm{bs}}_{\mathfrak{s}}(M_{0,\varepsilon})$ and $[\mathbf{tp}_{\mathfrak{s}}(a_{\gamma_j}, M_{0,j}, M_{\lambda j,j}) \in \mathscr{S}^{\mathrm{bs}}_{\mathfrak{s}}(M_{0,j}) \Rightarrow \mathbf{tp}_{\mathfrak{s}}(a_{\gamma_j}, M_{0,\varepsilon}, M_{\lambda j,\varepsilon})$ forks over $M_{0,j}]$ where $\mathscr{W}_j = \{\alpha \in \mathscr{U}_i$: we can find M', N' such that $\mathrm{WNF}(M_{0,j}, M_{\lambda j,j}, M', N')$ and $\mathbf{tp}_{\mathfrak{s}}(a_\alpha, M', N') \in \mathscr{S}^{\mathrm{bs}}_{\mathfrak{s}}(M')$ but is not a non-forking extension of $\mathbf{tp}_{\mathfrak{s}}(a, M_{0,j}, M_{\lambda j,j})$, e.g. $\mathbf{tp}_{\mathfrak{s}}(a, M_{0,j}, M_{\lambda j,j}) \notin \mathscr{S}^{\mathrm{bs}}_{\mathfrak{s}}(M_{0,j})\}$.

There is no problem to carry the definition.

Now by 7.22

$(*)_2$ $M_{\lambda\lambda,\lambda}$ is brimmed over $M_{0,\lambda}$.

So by $(*)_1 + (*)_2$ and $\mathfrak{s}$ being categorical without loss of generality

$(*)_3$ $(M_0, N_0) = (M_{0,\lambda}, M_{\lambda\lambda,\lambda})$.

Clearly $M_{\lambda\lambda,\lambda}$ is the union of the $\le_{\mathfrak{s}}$-increasing continuous chain $\langle M_{\lambda j,j} : j \le \lambda \rangle$ hence $j_1(*)$ is well defined where:

$(*)_4$ $j_1(*) = \min\{j < \lambda : c \in M_{\lambda j,j}\}$.

By clause (d) of $\circledast$ for some $j(*)$ we have

$(*)_5$ $j(*) \in [j_1(*), \lambda)$ and $c \in \{a_\alpha : \alpha \in \mathscr{U}_{j(*)}\}$.

So by the choice of $j(*), \gamma(*)$ is well defined where

$(*)_6$ $\gamma(*) = \min\{\gamma \in \mathscr{U}_{j(*)} : a_\gamma = c\} < \lambda$.

Note that

$(*)_7$ $(M_{0,j}, M_{\lambda j,j}) \le_{\rm wnf} (M_{0,\lambda}, M_{\lambda\lambda,\lambda}) = (M_0, N_0) \le_{\rm wnf} (M_1, N_1)$ for $j \in [j(*), \lambda)$.

Also

$(*)_8$ if $j \in [j(*), \lambda)$ then $\mathbf{tp}_{\mathfrak{s}}(c, M_{0,j}, M_{\lambda j,j}) \notin \mathscr{S}^{\rm bs}_{\mathfrak{s}}(M_{0,j})$.

[Why? By $(*)_7$, we have $\mathrm{WNF}_{\mathfrak{s}}(M_{0,j}, M_{\lambda j,j}, M_{0,\lambda}, M_{\lambda\lambda,\lambda})$ hence if $(\mathbf{tp}_{\mathfrak{s}}(c, M_{0,j}, M_{\lambda j,j}) \in \mathscr{S}^{\rm bs}_{\mathfrak{s}}(M_{0,j})$ then $(M_{0,j}, M_{\lambda j,j}, c) \in K^{3,\rm bs}_{\mathfrak{s}}$ but by $(*)_7$ we have $\mathrm{WNF}(M_{0,j}, M_{\lambda j,j}, M_1, N_1)$ and WNF respects $\mathfrak{s}$ hence $\mathbf{tp}_{\mathfrak{s}}(c, M_1, N_1)$ does not fork over $M_{0,j}$, so by monotonicity it does not fork over $M_{0,\lambda} = M_0$ and this contradicts $(*)_0$.]

$(*)_9$ if $j \in [j(*), \lambda)$ then $\min(\mathscr{W}_j) \le \gamma(*)$.

[Why? As $\gamma(*) \in \mathscr{W}_j$, i.e. satisfies the requirement which appear in clause (e) of $\circledast$ that is (M_1, N_1) here can stand for (M', N') there.]

So by cardinality considerations for some $j_1 < j_2$ from $[j(*), \lambda)$ we have $\min(\mathscr{W}_{j_1}) = \min(\mathscr{W}_{j_2})$ but this gives a contradiction as in the proof of $(*)_8$. $\square_{7.25}$

<u>7.26 Exercise</u>: Show that in Hypothesis 7.20 we can omit "$\mathfrak{s}$ is categorical (in λ)".

[<u>Hint</u>: The only place it is used is in showing $(*)_3$ during the proof of 7.25. To avoid it in $\circledast$ there waive clause (c) and add $j \le \lambda \Rightarrow M_{0,j} = M_0$.]

<u>7.27 Exercise</u>: Show that in this section we can replace clause (c) of Hypothesis 7.1, i.e. "$\mathfrak{s}$ is categorical in λ" by

$(c)'$ $\dot I(K_{\mathfrak{s}}) \le \lambda$
or just

$(c)''$ $\xi_{\mathfrak{s}} = \sup\{\xi_M : M \in K_{\mathfrak{s}}\} < \lambda^+$ where for $M \in K_{\mathfrak{s}}$ we let $\xi_N = \min\{\alpha_{\mathbf{d}}$: there is a $\mathfrak{u}$-free $(\alpha_{\mathbf{d}}, 0)$-rectangle such that $M^{\mathbf{d}}_{0,0} = M, (M^{\mathbf{d}}_{i,0}, M^{\mathbf{d}}_{i,0}, a^{\mathbf{d}}_{i,0}) \in K^{3,\rm up}_{\mathfrak{s}}$ and $M^{\mathbf{d}}_{\alpha(\mathbf{d}),0}$ is universal over $M^{\mathbf{d}}_{0,0} = M\}$.

[Hint: The only place we use "$\mathfrak{s}$ is categorical in λ" is in claim 7.14 more fully in 7.10(2) we would like to have a bound $< \lambda^+$ on $\alpha_{\mathbf{d}}$ not depending on M_0, see 7.11 (and then quoting it). It is used to define $\xi_{\mathfrak{s}}$. As $(c)' \Rightarrow (c)''$ without loss of generality we assume $(c)''$.]

<u>7.28 Exercise</u>: (Brimmed lifting, compare with IV.705-1.15K(4).)
1) For any $\le_{\mathfrak{s}}$-increasing continuous sequence $\langle M_\alpha : \alpha \le \alpha(*)\rangle$ with $\alpha(*) < \lambda^+_{\mathfrak{s}}$ we can find $\bar N$ such that

$\circledast$ (a) $\bar N = \langle N_\alpha : \alpha \le \alpha(*)\rangle$ is $\le_{\mathfrak{s}}$-increasing continuous

- (b) $\mathrm{WNF}(M_\alpha, N_\alpha, M_\beta, N_\beta)$ for $\alpha < \beta \le \alpha(*)$
- (c) N_α is brimmed over M_α for $\alpha = 0$ and moreover for every $\alpha \le \alpha(*)$
- (d) $N_{\alpha+1}$ is brimmed over $M_{\alpha+1} \cup N_\alpha$ for $\alpha < \alpha(*)$.

2) In fact the moreover in clause (c) follows from (a),(b),(d); an addition

$(c)^+$ N_β is brimmed over $N_\alpha \cup M_\beta$ for $\alpha < \beta \le \alpha(*)$.

[Hint: Similar to 5.19 or 5.20 or note that by 7.19 we can quote Chapter IV.]

§8 Density of $K^{3,\mathrm{uq}}_{\mathfrak{s}}$ for good λ-frames

We shall prove non-structure from failure of density for $K^{3,\mathrm{uq}}_{\mathfrak{s}}$ in two rounds. First, in 8.6 - 8.10 we prove the wnf-delayed version. Second, in 8.14 - 8.17 - we use its conclusion to prove the general case. Of course, by §6, we can assume as in §7:

8.1 Hypothesis. *We assume (after 8.2)*

- (a) $\mathfrak{s}$ *is an almost*[28] *good* λ*-frame*
- (b) WNF *is a weak non-forking relation on* $K_{\mathfrak{s}}$ *respecting* $\mathfrak{s}$ *with disjointness (not necessarily the one from Definition 7.3, but Hypothesis 7.20 holds).*

We can justify Hypothesis 8.1 by

8.2 Observation. Instead clause (b) of 8.1 we can assume

- $(b)'$ $\mathfrak{s}$ has existence for $K^{3,\mathrm{up}}_{\mathfrak{s},\lambda^+}$ (so we may use the consequences of Conclusion 7.17)
- (c) $\mathfrak{s}$ is categorical in λ.

Proof. Why? First note that Hypothesis 7.1 holds:

<u>Part (1)</u> there, $\mathfrak{s}$ is an almost good λ-frame, is clause (a) of Hypothesis 8.1.

<u>Part (2)</u> there is clause (b)$'$ assumed above.

<u>Part (3)</u>, categoricity in λ, there is clause (c) above.

<u>Part (4)</u> there, disjointness of $\mathfrak{s}$, holds by 5.23.

[28] by 7.17 clauses (b) + (c) implies that $\mathfrak{s}$ is actually a good λ-frame but we may ignore this

So the results of §7 holds, in particular the relation WNF := $\mathrm{WNF}_{\mathfrak{s}}$ defined in Definition 7.3 is a weak non-forking relation (on $K_{\mathfrak{s}}$) respecting $\mathfrak{s}$, by Claim 7.17. $\square_{8.2}$

We now use a relative of $\mathfrak{u}^1_{\mathfrak{s}}$ from Definition 4.29; this will be the default value of $\mathfrak{u}$ in this section so ∂ will be $\partial_{\mathfrak{u}} = \lambda^+$.

8.3 Definition. For $\mathfrak{s}$ as in 8.1 we define $\mathfrak{u} = \mathfrak{u}^3_{\mathfrak{s}}$ as follows:

(a) $\partial_{\mathfrak{u}} = \lambda^+ (= \lambda^+_{\mathfrak{s}})$

(b) $\mathfrak{K}_{\mathfrak{u}} = \mathfrak{K}_{\mathfrak{s}}$ (or $\mathfrak{K}'_{\mathfrak{s}}$ see 4.25, 4.26; but not necessary by (c) of 8.1)

(c) $\mathrm{FR}_\ell = \{(M, N, \mathbf{J}) : M \le_{\mathfrak{s}} N$ and $\mathbf{J} = \emptyset$ or $\mathbf{J} = \{a\}$ and $(M, N, a) \in K^{3,\mathrm{bs}}_{\mathfrak{s}}\}$

(d) $\le^\ell_{\mathfrak{u}}$ is defined by $(M_0, N_0, \mathbf{J}_0) \le^\ell_{\mathfrak{u}} (M_1, N_1, \mathbf{J}_1)$ <u>when</u>

  (α) $\mathrm{WNF}(M_0, N_0, M_1, N_1)$

  (β) $\mathbf{J}_1 \subseteq \mathbf{J}_0$

  (γ) if $\mathbf{J}_0 = \{a\}$ then $\mathbf{J}_1 = \{a\}$ hence $(M_0, N_0, a) \le^{\mathrm{bs}}_{\mathfrak{s}} (M_1, N_1, a)$ by "WNF respects $\mathfrak{s}$", see Hypothesis 8.1 and Definition 7.18; if we use $\mathrm{WNF}_{\mathfrak{s}}$ then we can quote 7.17(2), also by 7.5(1).

8.4 Remark. 0) The choice in 8.3 gives us symmetry, etc., i.e. $\mathfrak{u}$ is self-dual, this sometimes helps.
1) We could define $\mathrm{FR}_1, \le_1$ as above but

(e) $\mathrm{FR}_2 = \{(M, N, \mathbf{J}) : M \le_{\mathfrak{s}} N$ and $\mathbf{J} = \emptyset$ or $\mathbf{J} = \{a\}, (M, N, a) \in K^{3,\mathrm{bs}}_{\mathfrak{s}}\}$

(f) $\le_2$ is defined by:
$(M_0, N_0, \mathbf{J}_0) \le_1 (M_1, N_1, \mathbf{J}_1)$ <u>when</u> (both are from FR_1 and)

  (α) $M_0 \le_{\mathfrak{s}} M_1$ and $N_0 \le_{\mathfrak{s}} N_1$

  (β)′ $\mathbf{J}_0 \subseteq \mathbf{J}_1$

  (β)″ if $\mathbf{J}_0 = \{a\}$ then $\mathbf{J}_1 = \{a\}$ and $(M_0, N_0, a) \le^{\mathrm{bs}}_{\mathfrak{s}} (M_1, N_1, a)$.

2) We call it $\mathfrak{u}^{3,*}_{\mathfrak{s}}$. However, then for proving 8.14 we have to use $\mathfrak{u}^{*,3}_{\mathfrak{s}}$ which is defined similarly interchanging $(\mathrm{FR}_1, \le_1)$ with $(\mathrm{FR}_2, \le_2)$. Thus we lose "self-dual".

8.5 Claim. *1) $\mathfrak{u}$ is a nice construction framework which is self-dual.*
2) For almost$_2$ every $(\bar{M}, \bar{\mathbf{J}}, \mathbf{f}) \in K^{\mathrm{qt}}_{\mathfrak{s}}$ the model M_∂ is saturated above λ.
3) $\mathfrak{u}$ is monotonic (see 1.13(1)), hereditary (see 3.17(12)), hereditary for $=_*$ if $=_*$ is a fake equality for $\mathfrak{s}$ (see 4.25) and has interpolation (see 3.21).

Proof. 1) As in earlier cases (see 4.30(1)), 5.11(1)).
2) As in 4.30(2) or 5.11(2).
3) check. $\square_{8.5}$

8.6 Theorem. *We have $\dot{I}(\lambda^{++}, K^{\mathfrak{s}}(\lambda^+\text{-saturated})) \geq \mu_{\mathrm{unif}}(\lambda^{++}, 2^{\lambda^+})$ and even $\dot{I}(K^{\mathfrak{u},\mathfrak{h}}_{\lambda^{++}}) \geq \mu_{\mathrm{unif}}(\lambda^{++}, 2^{\lambda^+})$ for any $\mathfrak{u} - \{0,2\}$-appropriate function $\mathfrak{h}$ <u>when</u>:*

- (a) $2^\lambda < 2^{\lambda^+} < 2^{\lambda^{++}}$
- (b) *$\mathfrak{u}$ fails wnf-delayed uniqueness for* WNF *see Definition 8.7 below.*

Remark. Note that we have some versions of delayed uniqueness: the straight one, the one with WNF and the one in §5 and more.

Before we prove Theorem 8.6

8.7 Definition. We say that (the almost good λ-frame) $\mathfrak{s}$ has wnf-delayed uniqueness for WNF <u>when</u>: (if WNF is clear from the context we may omit it)

$\boxtimes$ for every $(M_0, N_0, a) \in K^{3,\mathrm{bs}}_{\mathfrak{s}}$ we can find (M_1, N_1) such that
- (a) $(M_0, N_0, a) \leq^1_{\mathfrak{u}} (M_1, N_1, a)$, i.e. $(M_0, N_0, a) \leq_{\mathrm{bs}} (M_1, N_1, a)$ and WNF(M_0, N_0, M_1, N_1), see clause (b) of Hypothesis 8.1 and
- (b) if $(M_1, N_1, a) \leq^1_{\mathfrak{u}} (M_\ell, N_\ell, a)$ hence WNF$(M_1, N_1, M_\ell, N_\ell)$ for $\ell = 2, 3$ and $M_2 = M_3$ <u>then</u> N_2, N_3 are $\leq_{\mathfrak{s}}$-compatible over $M_2 \cup N_0$, that is we can find a pair (f, N') such that
  - (α) $N_3 \leq_{\mathfrak{s}} N'$
  - (β) f is a $\leq_{\mathfrak{s}}$-embedding of N_2 ito N'
  - (γ) f is the identity on N_0 (not necessarily N_1!) and on $M_2 = M_3$.

Remark. A point of 8.7 is that we look for uniqueness among $\leq^1_{\mathfrak{u}}$-extensions (if 8.2 apply then $\leq_{\mathrm{wnf}}$-extensions) and, of course, it is "delayed", i.e. possibly $M_0 \neq M_1$.

8.8 Observation. In Definition 8.7 we can without loss of generality demand that M_1 is brimmed over M_0. Hence M_1 can be any pregiven $\le_{\mathfrak{s}}$-extension of M_0 brimmed over it such that $M_1 \cap N_0 = M_0$.

Proof. Read the definition. $\square_{8.8}$

8.9 Claim. *If $\mathfrak{s}$ (satisfies 8.1) and fails wnf-delayed uniqueness for* WNF *(i.e. satisfies 8.6(b))* then $\mathfrak{u} = \mathfrak{u}^3_{\mathfrak{s}}$ *has vertical coding, see Definition 2.9.*

Proof. Straight. $\square_{8.9}$

Proof of 8.6. Straight by the above and Theorem 2.11. $\square_{8.9}$

8.10 Remark. Note that the assumption of 8.9, failure of wnf-delayed uniqueness may suffice for a stronger version of 8.6 because given $\eta \in {}^{\partial^+}2$ and $(\bar{M}^\eta, \bar{\mathbf{J}}^\eta, \mathbf{f}^\eta) \in K^{\rm qt}_{\mathfrak{s}}$, we can find 2^∂ extensions $\langle(\bar{M}^{\eta,\rho}, \bar{\mathbf{J}}^{\eta,\rho}, \mathbf{f}^{\eta,\rho}) : \rho \in {}^\omega 2\rangle$ and $\alpha(*) < \partial$ such that $M^{\eta,\rho}_{\alpha(0)} = N_*$ and $\langle M^{\eta,\rho}_\partial : \rho \in {}^\partial 2\rangle$ are pairwise non-isomorphic over $M^\eta_\partial \cup N_*$. Does this help to omit the assumption $2^\lambda < 2^{\lambda^+}$?

8.11 Definition. 1) We say $\mathfrak{s}$ has uniqueness for WNF <u>when</u>:

if $\mathrm{WNF}(M^k_0, M^k_1, M^k_2, M^k_3)$ for $k = 1, 2$ and f_ℓ is an isomorphism from M^1_ℓ onto M^2_ℓ for $\ell = 0, 1, 2$ and $f_0 \subseteq f_1, f_0 \subseteq f_2$ <u>then</u> there is a pair (N, f) such that $M^2_3 \le_{\mathfrak{s}} N$ and f is a $\le_{\mathfrak{s}}$-embedding of M^1_3 into N extending $f_1 \cup f_2$.

2) We say $(M_0, M_1, a) \in K^{3,\rm bs}_{\mathfrak{s}}$ has non-uniqueness for WNF <u>when</u>: if $(M_0, M_1, a) \le^1_{\mathfrak{u}} (M'_0, M'_1, a)$ <u>then</u> we can find $\langle M^k_\ell : \ell \le 3, k = 1, 2\rangle, \langle f_\ell : \ell \le 2\rangle$ such that

$\circledast$ (a) $(M^k_0, M^k_1, a) \le^1_{\mathfrak{u}} (M^k_2, M^k_3, a)$ for $k = 1, 2$

(b) $M^k_0 = M'_0, M^k_1 = M'_1$ for $k = 1, 2$

(c) $M^1_2 = M^2_2$ and f_2 is the identity on M^2_1

(d) there is no pair (N, f) such that $M^2_3 \le_{\mathfrak{s}} N$ and f is a $\le_{\mathfrak{s}}$-embedding of M^1_3 into N extending $\mathrm{id}_{M^k_1} \cup \mathrm{id}_{M^k_2}$ (which does not depend on k).

3) We say that $\mathfrak{s}$ has non-uniqueness for WNF when some triple $(M_0, M_1, a) \in K^{3,\rm bs}_{\mathfrak{s}}$ has it.

8.12 Observation. 1) Assume $\mathfrak{s}$ is categorical (in λ). Then $\mathfrak{s}$ has non-uniqueness for WNF iff it does not have uniqueness for WNF iff $\mathfrak{s}$ fails existence for $K^{3,\mathrm{ur}}_{\mathfrak{s},\mathfrak{u}}$, see below.
2) If $\mathfrak{s}$ is categorical (in λ), has existence for $K^{3,\mathrm{up}}_{\mathfrak{s}} = K^{3,\mathrm{up}}_{\mathfrak{s},\lambda^+}$ and has uniqueness for WNF and 7.9 holds for WNF (i.e. if $(M_1, N_1, a) \in K^{3,\mathrm{up}}_{\mathfrak{s}}$ and $(M_1, N_1, a) \leq_{\mathrm{bs}} (M_2, N_2, a)$ then[29] $\mathrm{WNF}(M_1, M_1, N_1, N_2)$) then $\mathfrak{s}$ has existence for $K^{3,\mathrm{uq}}_{\mathfrak{s}}$ and $K^{3,\mathrm{up}}_{\mathfrak{s}} \subseteq K^{3,\mathrm{uq}}_{\mathfrak{s}}$.
3) $K^{3,\mathrm{uq}}_{\mathfrak{s}} \subseteq K^{3,\mathrm{up}}_{\mathfrak{s}}$.
4) If $\mathfrak{s}$ has uniqueness for WNF then WNF is a non-forking relation on $\mathfrak{K}_\lambda$ respecting $\mathfrak{s}$.

Remark. Note that 8.12(3) is used in §4(F).

8.13 Definition. 1) Let $K^{3,\mathrm{ur}}_{\mathfrak{s},\mathfrak{u}}$ be the class of triples $(M, N, a) \in K^{3,\mathrm{bs}}_{\mathfrak{s}}$ such that: if $(M, N, a) \leq^1_{\mathfrak{u}} (M', N'_\ell, a)$ for $\ell = 1, 2$ then we can find a pair (N^*, f) such that $N'_2 \leq_{\mathfrak{s}} N^*$ and f is a $\leq_{\mathfrak{s}}$-embedding of N'_1 into N^* extending $\mathrm{id}_N \cup \mathrm{id}_{M'}$.
2) $\mathfrak{s}$ has existence for $K^{3,\mathrm{ur}}_{\mathfrak{s}}$ when: if $M \in K_{\mathfrak{s}}$ and $p \in \mathscr{S}^{\mathrm{bs}}_{\mathfrak{s}}(M)$ then for some pair (N, a) the triple $(M, N, a) \in K^{3,\mathrm{ur}}_{\mathfrak{s}}$ realizes p.
3) If WNF is $\mathrm{WNF}_{\mathfrak{s}}$ and $\mathfrak{u}$ is defined as in 8.7 above then we may omit $\mathfrak{u}$.

Proof. 1) By the definition

(∗) if $\mathfrak{s}$ has non-uniqueness for WNF then $\mathfrak{s}$ does not have uniqueness for $\mathfrak{s}$.

Now

Case 1: $\mathfrak{s}$ fails existence for $K^{3,\mathrm{ur}}_{\mathfrak{s}}$.

We shall show that $\mathfrak{s}$ has the non-uniqueness property; this suffices by (∗). Let (M, p) exemplify it and let (N, a) be such that $(M, N, a) \in K^{3,\mathrm{bs}}_{\mathfrak{s}}$ realizes p and we shall prove that (M, N, a) is as required in Definition 8.11(2).

Let $(M', N', a) \in K^{3,\mathrm{bs}}_{\mathfrak{s}}$ be $\leq^1_{\mathfrak{u}}$-above (M, N, a). We can find (f_*, N_*, a_*) such that $(M, N_*, a_*) \in K^{3,\mathrm{bs}}_{\mathfrak{s}}$ realizes p and f_* is an isomorphism from N' onto N_* which maps M' onto M and a to a_*. [Why it exists? See 5.20 recalling $\mathfrak{s}$ is categorical.] So $(M, N_*, a_*) \in K^{3,\mathrm{bs}}_{\mathfrak{s}}$ and $\mathbf{tp}(a^*, M, N_*) = p$ hence by the choice of M and p clearly $(M, N_*, a_*) \notin K^{3,\mathrm{ur}}_{\mathfrak{s},\mathfrak{u}}$ so by Definition 8.13(1) we can find M', N_1, N_2 as there such that there are no (N^*, f) as there. But this means that for (M, N_*, a_*) we can

[29] so really WNF = $\mathrm{WNF}_{\mathfrak{s}}$

find $\langle M^k_\ell : \ell \le 3, k = 1,2\rangle$ as required in 8.11(2). By chasing maps this holds also for (M', N', a) so we are done.

Case 2: $\mathfrak{s}$ has existence for $K^{3,\mathrm{ur}}_{\mathfrak{s}}$.

We shall show that $\mathfrak{s}$ has uniqueness for WNF. This suffices by $(*)$.

First note

$\boxtimes$ if $(*)_1$ and $(*)_2$ below then M^1_3, M^2_3 are isomorphic for $M_2 \cup M_1$ when:

$(*)_1$ (a) $\bar M = \langle M_{0,\alpha} : \alpha \le \alpha(*)\rangle$ is $<_{\mathfrak{s}}$-increasing continuous

(b) $(M_{0,\alpha}, M_{0,\alpha+1}, a_\alpha) \in K^{3,\mathrm{ur}}_{\mathfrak{s},\mathfrak{u}}$ for $\alpha < \alpha(*)$

(c) $M_{0,\alpha(*)}$ is brimmed over $M_{0,0}$

$(*)_2$ (α) $\mathrm{WNF}(M_0, M_1, M_2, M^k_3)$ for $k = 1,2$

(β) $M_0 = M_{0,0}$ and $M_1 = M_{0,\delta}$

(γ) M^k_3 is brimmed over $M_1 \cup M_2$.

[Why? As in previous arguments in §7, we lift $\bar M$ by $(*)_2(\alpha)$ and clause (g) of Definition 7.18 of "WNF is a weak non-forking relation on $\mathfrak{K}_{\mathfrak{s}}$", i.e. being as in the proof of 7.17 and then use $(*)_1(b)$.]

Next

$\boxplus$ we can weaken $(*)_2(\beta)$ to $M_{0,0} = M_0 \le_{\mathfrak{s}} M_1 \le_{\mathfrak{s}} M_{0,\alpha(*)}$.

[By the properties of WNF.]

Checking the definitions we are done recalling 7.10(2B), or pedantically repeating its proof to get $\langle M_{0,i} : i \le \alpha(*)\rangle$ as in $(*)_1$.

2) We are assuming that $\mathfrak{s}$ has existence for $K^{3,\mathrm{up}}_{\mathfrak{s}}$ so it suffices to prove $K^{3,\mathrm{up}}_{\mathfrak{s}} \subseteq K^{3,\mathrm{uq}}_{\mathfrak{s}}$. So assume $(M_0, N_0, a) \in K^{3,\mathrm{up}}_{\mathfrak{s}}$ and $(M_0, N_0, a) \le_{\mathrm{bs}} (M_\ell, N_\ell, a)$ for $\ell = 2,3$ and $M_2 = M_3$. By 7.9 it follows that $\mathrm{WNF}(M_0, N_0, M_\ell, N_\ell)$ so by Definition 8.3 we have $(M_0, N_0, a) \le^1_{\mathfrak{u}} (M_\ell, N_\ell, a)$. Applying Definition 8.11(1) we are done.

3),4) Clear by the definitions. $\square_{8.12}$

* * *

8.14 Theorem. $\dot I(\lambda^{++}, K^{\mathfrak{s}}) \ge \dot I(\lambda^{++}, K^{\mathfrak{s}}(\lambda^+\text{-saturated})) \ge \mu_{\mathrm{unif}}(\lambda^{++}, 2^{\lambda^+})$ *and even* $\dot I(K^{\mathfrak{u},\mathfrak{h}}_{\lambda^{++}}) \ge \mu_{\mathrm{unif}}(\lambda^{++}, 2^{\lambda^+})$ *for any* $\mathfrak{u}_{\mathfrak{s}}-\{0,2\}$*-appropriate* $\mathfrak{h}$ *(so we can restrict ourselves to models* λ^+*-saturated above* λ *and if* $\mathfrak{s} = \mathfrak{s}'$ *also to* $\tau_{\mathfrak{s}}$*-fuller ones)* *when:*

(a) $2^\lambda < 2^{\lambda^+} < 2^{\lambda^+}$

(b) $\mathfrak{s}$ *has non-uniqueness for* WNF *(for every* $M \in K_{\mathfrak{s}}$*)*

(c) $\mathfrak{s}$ *has* wnf-*delayed uniqueness for* WNF.

Proof. We first prove claim 8.16.

Note that proving them we can use freely 8.7, 8.3, 8.8 and that wnf-delayed uniqueness replaces the use of 10.7.

<u>8.15 Explanation</u>: As $\mathrm{FR}_1^{\mathfrak{u}} = \mathrm{FR}_2^{\mathfrak{u}}$ there is symmetry, i.e. $\mathfrak{u}$ is self-dual. The wnf-delayed uniqueness was gotten vertically, i.e. from its failure we got a non-structure result (8.6) relying on vertical coding, i.e. 2.11. But now we shall use it horizontally; we shall construct over $(\bar M, \bar{\mathbf{J}}, \mathbf{f})$ with $M_\partial \in K^{\mathfrak{s}}_{\lambda^+}$ saturated above λ, a tree $\langle (M^\rho, \bar{\mathbf{J}}^\rho, \mathbf{f}^\ell) : \rho \in {}^{\partial \geq}2\rangle$ as in weak coding but each is not as usual but a sequence of length $\ell g(\rho)$ such extensions. In fact we use the λ-wide case of §10, i.e. 10.14, 10.15 <u>without</u> quoting. So the "non-structure" is done in the "immediate successor" of $(\bar M, \bar{\mathbf{J}}, \mathbf{f})$. The rest of the section is intended to make the rest of the construction, in the ∂^+-direction, irrelevant (well, mod $\mathscr{D}_\partial$, etc) using the wnf-delayed uniqueness assumed in clause (c) of 8.14, justified by 8.6. The net result is that we can find $\langle (\bar M^\rho, \bar{\mathbf{J}}^\rho, \mathbf{f}^\rho) : \rho \in {}^\partial 2\rangle$ which are $\leq^{\mathrm{qt}}_{\mathfrak{u}}$-above $(\bar M, \bar{\mathbf{J}}, \mathbf{f})$ and for $\rho \neq \nu \in {}^\partial 2$, there is no $\leq_{\mathfrak{K}}$-embedding of M^ρ_∂ into M'_∂ if $(\bar M^\nu, \mathbf{J}^\nu, \mathbf{f}^\nu) \leq^{\mathrm{qs}}_{\mathfrak{u}} (\bar M', \bar{\mathbf{J}}', \mathbf{f}')$. That comes instead of using $\bar{\mathbb{F}}$, the amalgamation choice functions in §10.

For constructing $\langle (\bar M^\rho, \bar{\mathbf{J}}^\rho, \mathbf{f}^\rho) : \rho \in {}^\partial 2\rangle$ as above, again we use $\langle (\bar M^{\rho,\alpha}, \bar{\mathbf{J}}^{\rho,\alpha}, \mathbf{f}^{\rho,\alpha}) : \rho \in {}^i 2, \alpha < \lambda\rangle$ for $i \leq \partial$ such that for a club of $\delta < \partial$ the model $\cup\{M^{\rho,\alpha}_\delta : \alpha < \delta\}$ is brimmed over $M^{\rho,\beta}$ for $\beta < \gamma$.

8.16 Claim. *[Under the assumptions of 8.14]*
If $\boxtimes$ *then* $\circledast$ *where*

- $\boxtimes$ (a) $(M, N, a) \in K^{3,\mathrm{bs}}_{\mathfrak{s}}$ *has non-uniqueness for* $\mathrm{WNF}_{\mathfrak{s}}$
  - (b) $\delta < \lambda^+$ *is divisible by* λ^3
  - (c) $\mathbf{d}$ *is a* $\mathfrak{u}$*-free* $(0, \delta)$*-rectangle, let* $M_\alpha = M^{\mathbf{d}}_{0,\alpha}$ *for* $\alpha \leq \delta, a_\alpha = a^{\mathbf{d}}_{0,\alpha}$ *for* $\alpha < \delta$ *and* $a = a_0$
  - (d) $(M, N, a) \leq_{\mathrm{wnf}} (M_0, M_1, a)$ *equivalently* $(M, N, \{a\}) \leq^1_{\mathfrak{u}} (M_0, M_1, \mathbf{I}^{\mathbf{d}}_{0,0})$
  - (e) M_δ *is brimmed over* M_α *for* $\alpha < \delta$
  - (f) $\delta \in \mathrm{correct}(\langle M_\alpha : \alpha \leq \delta\rangle)$*, i.e. if* $M_\delta <_{\mathfrak{s}} N$ <u>*then*</u> *some* $p \in \mathscr{S}^{\mathrm{bs}}_{\mathfrak{s}}(M_\delta)$ *is realized in* N *and does not fork over* M_β *for some* $\beta < \delta$ *(on correctness, see Definition 5.14)*
  - (g) $(M_0, M', b) \in \mathrm{FR}_2$ *and* $M' \cap M_\delta = M_0$ *and* M' *is brimmed over* M_0
- $\circledast$ *there are* $\mathbf{d}_1, \mathbf{d}_2$ *such that*
  - (α) $\mathbf{d}_\ell$ *is a* $\mathfrak{u}$*-free* $(1, \delta + 1)$*-rectangle for* $\ell = 1, 2$
  - (β) $b^{\mathbf{d}}_{0,0} = b$

(γ) $\mathbf{d}_\ell \restriction (0,\delta) = \mathbf{d}$ *for* $\ell = 1,2$

(δ) $\mathbf{d}_1 \restriction (1,0) = \mathbf{d}_2 \restriction (1,0)$

(ε) $M^{\mathbf{d}_1}_{1,0} = M^{\mathbf{d}_2}_{1,0} = M'$ *and* $b^{\mathbf{d}_1}_{0,0} = b = b^{\mathbf{d}_2}_{0,0}$

(ζ) $M^{\mathbf{d}_1}_{\alpha,\delta}, M^{\mathbf{d}_2}_{\alpha,\delta}$ *are* $\tau_{\mathfrak{s}}$*-incompatible over* $(M^{\mathbf{d}_1}_{1,0} = M^{\mathbf{d}_2}_{1,0}) + M_1$

(η) *if* $k, \mathbf{d}_1, \mathbf{d}_2, f$ *satisfies* $\bullet_1 - \bullet_4$ *below, then we can find a triple* (g, N_1, N_2) *such that* $\ell = 1,2 \Rightarrow M^{\mathbf{d}_\ell}_{1,\alpha(\mathbf{d}_\ell)} \leq_{\mathfrak{s}} N_\ell$ *and* g *is an isomorphism from* N_1 *onto* N_2 *extending* $id_{M'} \cup f$ *where (for* $\ell = 1,2$*):*

- $\bullet_1$ $\mathbf{d}_\ell$ *is a* $\mathfrak{u}$*-free rectangle*
- $\bullet_2$ $\beta(\mathbf{d}_\ell) = 1$
- $\bullet_3$ $\alpha(\mathbf{d}_\ell) \geq \delta$ *and* $\mathbf{d}_\ell \restriction (0,\delta) = \mathbf{d}$
- $\bullet_4$ f *is an isomorphism from* $M^{\mathbf{d}_1}_{0,\alpha(\mathbf{d}_1)}$ *onto* $M^{\mathbf{d}_2}_{0,\alpha(\mathbf{d}_2)}$ *over* $M^{\mathbf{d}}_{0,\delta}$.

Remark. 1) In the proof we use wnf-delayed uniqueness.
2) This claim helps.

Proof. First, letting $M_* = M'$, we can choose M^1_*, M^2_* such that (for $\ell = 1,2$)

- $\circledast_1$ $M_* \leq_{\mathfrak{s}} M^\ell_*$
- $\circledast_2$ $M_1 \leq_{\mathfrak{s}} M^\ell_*$
- $\circledast_3$ $M^\ell_* \cap M_\delta = M_1$
- $\circledast_4$ $\mathrm{WNF}(M_0, M_1, M_*, M^\ell_*)$ hence $(M_0, M_*, b) \leq_{\mathrm{bs}} (M_1, M^\ell_*, b)$
- $\circledast_5$ M^1_*, M^2_* are τ-incompatible over $M_* + M_1$.

[Why? By $\boxtimes(a)$ and $\boxtimes(g)$ recalling Definition 8.11(2).]

Second, we choose N^ℓ_* for $\ell = 1,2$ such that

- $\circledast_6$ $\mathrm{WNF}(M_1, M_\delta, M^\ell_*, N^\ell_*)$ for $\ell = 1,2$
- $\circledast_7$ wnf-delayed uniqueness: if $\ell \in \{1,2\}$ and $M_{\delta+1}, N^1_{\delta+1}, N^2_{\delta+1}$ satisfies $\mathrm{WNF}(M_\delta, M_{\delta+1}, N^\ell_*, N^k_{\delta+1})$ for $k = 1,2$ <u>then</u> we can find (f, N) such that $N^2_{\delta+1} \leq_{\mathfrak{s}} N$ and f is a $\leq_{\mathfrak{s}}$-embedding of $N^1_{\delta+1}$ into N over M^1_* (hence over M_*) and over $M_{\delta+1}$.

[Why is this possible? As M_δ is brimmed over M_1 by cluase (e) of $\boxtimes$ we are assuming, and $\mathfrak{s}$ has wnf-delayed uniqueness by clause (c) of Theorem 8.14 and we apply it $(M_1, M^\ell_*, b) \le_2 (M_\delta, N^\ell_*, b)$ recalling $\mathfrak{u}$ is self-dual and 8.8.]

Note that in $\circledast_6$ we can replace N^ℓ_* by N^ℓ_{**} if $N^\ell_* \le_{\mathfrak{s}} N^\ell_{**}$ or $M^\ell_* \cup M_\delta \subseteq N^\ell_{**} \le_{\mathfrak{s}} N^\ell_*$.

Third, by the properties of WNF, for $\ell = 1, 2$ we can choose N^ℓ_{**} and a $\mathfrak{u}$-free $(1,\delta)$-rectangle $\mathbf{d}'_\ell$ with $M^{\mathbf{d}'_\ell}_{1,0} = M^\ell_*, b^{\mathbf{d}_\ell}_{0,0} = b, \mathbf{d}'_\ell \restriction (0,\delta) = \mathbf{d} \restriction ([0,0],[1,\delta])$ and $M^{\mathbf{d}'_\delta}_{1,\delta} \le_{\mathfrak{s}} N^\ell_{**}, N^\ell_* \le_{\mathfrak{s}} N^\ell_{**}$.

Now $\mathbf{d}'_1, \mathbf{d}'_2$ are as required. $\square_{8.16}$

8.17 Proof of 8.14. In this case, for variety, instead of using a theorem on $\mathfrak{u}$ from §2 or §3, we do it directly (except quoting 9.1). We fix a stationary $S \subseteq \partial$ such that $\partial \backslash S \notin$ WDmId(∂) and S is a set of limit ordinals.

We choose $\mathfrak{g}$ witnessing 8.5(2) for S so without loss of generality $S_{\mathfrak{g}} = S$ so $\mathfrak{g}$ is $\mathfrak{u}$-2-appropriate. Let $\mathfrak{h}$ be any $\mathfrak{u} - \{0,2\}$-appropriate function. We restrict ourselves to $K^{\mathrm{qt},*}_{\mathfrak{u}} := \{(\bar M, \bar{\mathbf{J}}, \mathbf{f}) \in K^{\mathrm{qt}}_{\mathfrak{u}} : M_\partial \in K^{\mathfrak{s}}_{\lambda^+}$ is saturated (above λ), M_∂ has universe an ordinal $< \partial^+$ and $\mathbf{f} \restriction (\partial \backslash S)$ is constantly 1 and $(\alpha \in \partial \backslash S \Rightarrow M_{\alpha+1}$ is brimmed over $M_\alpha\}$. We now choose $\langle (\bar M^\eta, \bar{\mathbf{J}}^\eta, \mathbf{f}^\eta) : \eta \in {}^\alpha(2^\partial)\rangle$ by induction on $\alpha < \partial^+$ such that

$\oplus_\alpha$ (a) $(\bar M^\eta, \bar{\mathbf{J}}^\eta, \mathbf{f}^\eta) \in K^{\mathrm{qt},*}_{\mathfrak{u}}$ for $\eta \in {}^\alpha(2^\partial)$

(b) $\langle (\bar M^{\eta\restriction\beta}, \bar{\mathbf{J}}^{\eta\restriction\beta}, \mathbf{f}^{\eta\restriction\beta}) : \beta \le \alpha\rangle$ is $\le^{\mathrm{qt}}_{\mathfrak{u}}$-increasing continuous

(c) if $\alpha = \beta+1$ and β is non-limit and $\eta \in {}^\alpha(2^\partial)$ then the pair $((\bar M^{\eta\restriction\beta}, \bar{\mathbf{J}}^{\eta\restriction\beta}, \mathbf{f}^{\eta\restriction\beta}), (\bar M^\eta, \bar{\mathbf{J}}^\eta, \mathbf{f}^\eta))$ obeys[30] $\mathfrak{g}$ if α is even and obeys $\mathfrak{h}$ if α is odd

(d) if $\alpha = \beta + 1, \beta$ is a limit ordinal, $\nu \in {}^\beta(2^\partial)$ and $\varepsilon_1 \ne \varepsilon_2 < 2^\partial$ and so $\eta^\ell = \nu\hat{\ }\langle\varepsilon_\ell\rangle$ is from ${}^\alpha(2^\partial)$ for $\ell = 1,2$ <u>then</u> not only $M^{\eta^1}_\partial, M^{\eta^2}_\partial$ are not isomorphic over M^ν_∂, but if $(\bar M^{\eta^\ell}, \bar{\mathbf{J}}^{\eta^\ell}, \mathbf{f}^{\eta^\ell}) \le^{\mathrm{qt}}_{\mathfrak{u}} (\bar M^\ell, \bar{\mathbf{J}}^\ell, \mathbf{f}^\ell)$ for $\ell = 1,2$ then $M^1_\partial, M^2_\partial$ are not isomorphic over M^ν_∂.

By 9.1 this suffices. For $\alpha = 0$ and α limit there are no problems (well we have to show that the limit exists which hold by 1.19(4), and belongs to $K^{\mathrm{qt},*}_{\mathfrak{u}}$, but this is easy by 7.28(2)).

So assume $\alpha = \beta + 1, \eta \in {}^\beta 2$ and we should choose $\langle (\bar M^{\eta\hat{\ }<\varepsilon>}, \bar{\mathbf{J}}^{\eta\hat{\ }<\varepsilon>}, \mathbf{f}^{\eta\hat{\ }<\varepsilon>}) : \varepsilon < 2^\partial\rangle$, let γ_* be the universe of M^η_∂.

Let E_1 be a club of $\partial = \lambda^+$ such that if $\alpha < \delta \in E_1$ then $\mathbf{f}(\alpha) < \delta$ and M^η_δ is brimmed over M^η_α. Let $E_2 = E_1 \cup \{[\delta, \delta + \mathbf{f}(\delta)] : \delta \in S \cap E_1\}$, and without loss of generality M^η_0 is brimmed and if $\delta \in S \cap E_1$ then $M_{\delta+1}$ is brimmed over M_δ (can use $\mathfrak{g}$ to guarantee this, or increase it inside M^η_γ with no harm). Let h be

[30] we may combine

the increasing continuous function from λ^+ onto E_2 and $E = \{\delta < \lambda^+ : \delta$ a limit ordinal and $h(\delta) = \delta\}$ a club of $\lambda^+ = \partial$.

So

$\boxplus$ (a) $(M^\eta_\alpha, M^\eta_{\alpha+1}, \mathbf{J}^\eta_\alpha) \in \mathrm{FR}^2_{\mathfrak{u}} = \mathrm{FR}^1_{\mathfrak{u}}$
(b) M^η_θ is brimmed
(c) $M^\eta_{h(\alpha+1)}$ is brimmed over $M^\eta_{h(\alpha)}$ if $\alpha \in E_1 \cap S$
(d) if $h(\alpha) \in S$ then $h(\alpha+1) = h(\alpha)+1$ (used?).

Now $\mathfrak{s}$ has non-uniqueness for WNF hence we can find (N, a) such that the triple (M^η_0, N, a) has the non-uniqueness property for WNF; without loss of generality $N \backslash M^\eta_0$ is $[\gamma_*, \gamma_* + i_{<>})$ for some ordinal $i_{<>} \le \lambda$.

Now we choose $\mathbf{d}_\rho$ for $\rho \in {}^\varepsilon 2$ by induction on $\varepsilon < \lambda^+$ such that (recalling $\mathfrak{u}$ is self-dual; note that $\mathbf{d}$ looks inverted letting $\bar{\alpha}^\varepsilon = \langle \lambda(1+\zeta) : \zeta \le \varepsilon \rangle$

$\odot$ for $\rho \in {}^\varepsilon 2$
(a) $\mathbf{d}_\rho$ is an $\mathfrak{u}$-free $(\bar{\alpha}^\varepsilon, \varepsilon)$-triangle
(b) $(M^\eta_0, N, \{a\}) = (M^{\mathbf{d}_\rho}_{0,0}, M^{\mathbf{d}_\rho}_{1,0}, \mathbf{J}^{\mathbf{d}_\rho}_{0,0})$
(c) if $\zeta < \varepsilon$ then $\mathbf{d}_{\rho \restriction \zeta} = \mathbf{d}_\rho \restriction (\bar{\alpha}^\zeta, \zeta)$
(d) $M^{\mathbf{d}_\rho}_{0,\zeta} = M^\eta_{h(\zeta)}$ for $\zeta \le \varepsilon$ and $\mathbf{I}^{\mathbf{d}_\rho}_{0,\zeta} = \mathbf{J}^\eta_\zeta$ for $\zeta < \varepsilon$
(e) $M^{\mathbf{d}_\rho}_{i+1,\zeta+1}$ is brimmed over $M^{\mathbf{d}_\rho}_{i,\zeta+1} \cup M^{\mathbf{d}_\rho}_{i,\zeta+1}$ when $\zeta < \varepsilon, i < \lambda(1+\zeta)$
(f) if $\varepsilon = \zeta + 1$ and $i < \lambda$ then[31] $M^{\mathbf{d}_\varepsilon}_{\lambda\varepsilon+i+1,\varepsilon}$ is brimmed over $M^{\mathbf{d}_\rho}_{\lambda\varepsilon+i,\varepsilon}$
(g) if $\varepsilon = \zeta + 1$ and $i < \lambda$ and $p \in \mathscr{S}^{\mathrm{bs}}_{\mathfrak{s}}(M^{\mathbf{d}_\varepsilon}_{\lambda\varepsilon+i,\varepsilon})$ <u>then</u> for λ ordinals $j \in [i, \lambda)$, the non-forking extension of p in $\mathscr{S}^{\mathrm{bs}}_{\mathfrak{s}}(M^{\mathbf{d}_\varepsilon}_{\lambda\varepsilon+j,\varepsilon})$ is realized by the $b \in \mathbf{J}^{\mathbf{d}_\varepsilon}_{\lambda\varepsilon+1,\varepsilon}$ in $M^{\mathbf{d}_\varepsilon}_{\lambda\varepsilon+j+1,\varepsilon}$
(h) if $\varepsilon \in E_! \cap S$ so $1 + \varepsilon = \varepsilon$ then clause (η) of $\circledast$ of 8.16 holds with dual$(\mathbf{d}_\rho \restriction (\lambda\varepsilon, 0))$, $\lambda\varepsilon$, (M^η_0, N, a), $(M^\eta_\varepsilon, M^\eta_{\varepsilon+1}, \mathbf{J}^\eta_\varepsilon)$, dual$(\mathbf{d}_{\rho\char`^<0>} \restriction [0, \lambda\varepsilon], [\varepsilon, \varepsilon+1])$, dual$(\mathbf{d}_{\rho\char`^<1>} \restriction [0, \lambda\varepsilon], [\varepsilon, \varepsilon+1))$ here standing for $\mathbf{d}$, δ, (M, N, a), $(M_0, M', \{b\})$, $\mathbf{d}_1$, $\mathbf{d}_2$ there
(i) the set $M^{\mathbf{d}_\rho}_{\lambda(1+\varepsilon),\varepsilon} \backslash M^\eta_\varepsilon$ is $[\gamma_*, \gamma_* + \lambda(1+\varepsilon))$.

There is no problem to carry the definition.

Lastly for $\rho \in {}^\partial 2$ we define $(\bar{M}^{\eta,\rho}, \bar{\mathbf{J}}^{\eta,\rho}, \mathbf{f}^{\eta,\rho})$ by: (let $E \subseteq \partial = \lambda^+$ be a thin enough club):

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[31] actually can waive clause (f),(g)

$\boxtimes$ (a) $M^{\eta,\rho}_\varepsilon = M^{\mathbf{d}_{\rho\restriction\varepsilon}}_{\lambda(1+\varepsilon),\varepsilon}$ for $\varepsilon < \lambda$

(b) $\mathbf{f}^{\eta,\rho} = \mathbf{f}^\eta$

(c) $\mathbf{J}^{\eta,\rho}_\varepsilon = \mathbf{J}^\eta_\varepsilon$ when $\varepsilon \in \cup\{[\delta, \delta + \mathbf{f}^\eta(\delta)) : \delta \in E\}$.

Now let $\langle S_\varepsilon : \varepsilon < \partial = \lambda^+\rangle$ be a partition of $\lambda^+ \backslash S$ to (pairwise disjoint) sets from $(\mathrm{WDmId}_{\lambda^+})^+$.

Now we define a function $\mathbf{c}$:

$(*)_1$ its domain is the set of $\mathbf{x} = (\rho_1, \rho_2, f, \mathbf{d})$ such that: for some $\varepsilon \in S \cap E \subseteq \lambda^+$

(a) $\rho_1, \rho_2 \in {}^\varepsilon 2$

(b) $\mathbf{d}$ is a $\mathfrak{u}$-free $(\varepsilon, 1)$-rectangle with $\mathbf{I}^{\mathbf{d}}_{\zeta,0} = \emptyset$ for $\zeta \le \varepsilon$

(c) $M^{\mathbf{d}}_{\zeta,0} = M^{\mathbf{d}_{\rho_2}}_{\lambda(1+\zeta),\zeta}$ for $\zeta \le \varepsilon$

(d) $\mathbf{J}^{\mathbf{d}}_{\zeta,0} = \mathbf{I}^{\mathbf{d}_{\rho_2}}_{0,\zeta}$ for $\zeta \le \varepsilon$

(e) $M^{\mathbf{d}}_{\varepsilon,1} \backslash M^{\mathbf{d}_{\rho_2}}_{\lambda(1+\varepsilon),\varepsilon} \subseteq [\gamma_* + \lambda^+, \gamma_* + \lambda^+ + \lambda^+)$

(f) f is a $\le_{\mathfrak{s}}$-embedding of $M^{\mathbf{d}_{\rho_1}}_{\lambda(1+\varepsilon),\varepsilon}$ into $M^{\mathbf{d}}_{\varepsilon,1}$ over $M^\eta_{h(\varepsilon)} = M^{\mathbf{d}_{\rho_\ell}}_{0,\varepsilon}$ for $\ell = 1, 2$

$(*)_2$ for $\mathbf{x} = (\rho_1, \rho_2, f, \mathbf{d})$ as above, say with $\varepsilon = \ell g(\rho_1) = \ell g(\rho_2)$ we have $\mathbf{c}(\mathbf{x}) = 1$ iff there are $\mathbf{d}^+, f^+, N^*$ such that letting $\nu_\ell = \rho_\ell \char`^ \langle 0\rangle$ for $\ell = 1, 2$:

(a) $\mathbf{d}^+$ is a $\mathfrak{u}$-free $(\varepsilon + 1, 1)$-rectangle

(b) $\mathbf{d}^+ \restriction (\varepsilon, 1) = \mathbf{d}$

(c) $M^{\mathbf{d}_{\nu_2}}_{\lambda(1+\varepsilon),\varepsilon+1} \le_{\mathfrak{s}} N^*$ and $M^{\mathbf{d}^+}_{\varepsilon+1,1} \le_{\mathfrak{s}} N^*$

(d) f^+ is a $\le_{\mathfrak{s}}$-embedding of $M^{\mathbf{d}_{\nu_1}}_{\lambda(1+\varepsilon),\varepsilon+1}$ into N^* extending $f \cup \mathrm{id}_{M^\eta_{h(\varepsilon)+1}}$.

Note

$\circledast$ if $(\bar M^{\eta,\rho}, \bar{\mathbf{J}}^{\eta,\ell}, \mathbf{f}^{\eta,\rho}) \le_{\mathrm{qt}} (\bar M^*, \mathbf{J}^*, \mathbf{f}^*)$ and the universe of M^*_∂ is an ordinal $< \lambda^{++}$ and π is a one-to-one mapping from M^*_∂ onto $\gamma_* + \partial + \partial$ over $\gamma_* + \partial$, then we can find $\langle \mathbf{e}_\varepsilon : \varepsilon < \lambda^+\rangle$ such that for some club E of ∂

(a) $\mathbf{e}_\varepsilon$ is a $\mathfrak{u}$-free $(\varepsilon, 1)$-rectangle

(b) $M^{\mathbf{e}_\varepsilon}_{\zeta,0} = M^{\mathbf{d}_\rho}_{\lambda(1+\zeta),\zeta}$ for $\zeta \le \varepsilon$

(c) $\mathbf{J}^{\mathbf{e}_\varepsilon}_{\zeta,0} = \mathbf{I}^{\mathbf{d}_\rho}_{\zeta,\zeta}$ for $\zeta < \varepsilon$

(d) $M^{\mathbf{e}_\varepsilon}_{\varepsilon,1} \backslash M^{\mathbf{d}_\rho}_{\lambda(1+\varepsilon),\varepsilon} \subseteq [\gamma_* + \lambda^+, \gamma_* + \lambda^+ + \lambda^+)$

(e) if $\varepsilon \in E$ then there is a $\mathfrak{u}$-free $(\varepsilon+1,1)$-rectangle $\mathbf{e}^+_\varepsilon$ such that $\mathbf{e}^+_\varepsilon \restriction (\varepsilon+1) = \mathbf{e}_{\varepsilon+1}$

(f) $\langle M^{\mathbf{e}_\varepsilon}_{\varepsilon,1} : \varepsilon < \partial\rangle$ is a $\leq_{\mathfrak{s}}$-increasing continuous sequence with union $\pi(M^*_\partial)$ which has universe $\gamma_* + \lambda^+ + \lambda^+$.

For each $\zeta < \lambda^+$ as $S_\zeta \subseteq \lambda^+$ does not belong to the weak dimaond ideal, there is a sequence $\varrho_\zeta \in {}^{(S_\zeta)}2$ such that

$(*)_3$ for any $\rho_1, \rho_2 \in {}^\partial 2$ such that $\rho_1 \restriction S_\zeta = 0_{S_\zeta}$ and $N_* \in K^{\mathfrak{s}}_{\lambda^+}, (\bar M^*, \bar{\mathbf{J}}^*, \mathbf{f}^*) \in K^{\rm qs}_{\mathfrak{s}}$ which is $\leq_{\rm qt}$-above $(\bar M^{\eta,\rho_2}, \bar{\mathbf{J}}^{\eta,\rho_2}, \mathbf{f}^{\eta,\rho_2})$ and $|M^*_\partial| = \gamma_{**} < \partial^+$ and $\leq_{\mathfrak{K}[\mathfrak{s}]}$-embedding f of M^{η,ρ_1}_∂ into M^*_∂ over M^η_∂, letting $\langle \mathbf{e}_\delta : \delta < \partial\rangle, \pi$ be as in $\circledast$ for the pair $((\bar M^\eta, \bar{\mathbf{J}}^\eta, \mathbf{f}^\eta), (\bar M^*, \bar{\mathbf{J}}^*, \mathbf{f}^*)$ the set $\{\delta \in S_\varepsilon : \mathbf{c}(\rho_1 \restriction \delta, \rho_2 \restriction \delta, f \restriction \delta, \mathbf{e}_\delta) = \varrho(\delta)\}$ is stationary.

Now for any $u \subseteq \partial$ we define $\rho_u \in {}^\lambda 2$ by

$(*)_4$ for $\zeta < \partial, \ell < 2$ let $\rho_u \restriction S_{2\zeta+\ell}$ be $0_{2\zeta+\ell}$ if $[\zeta \notin u \leftrightarrow \ell = 0]$ and ϱ_ζ otherwise

$(*)_5$ let $\rho_u \restriction S_0$ be constantly zero.

Let $\langle u(\alpha) : \alpha < 2^\partial\rangle$ list $\mathscr{P}(\gamma)$ and for $\alpha < 2^\partial$ let $(\bar M^{\eta\hat{}<\alpha>}, \mathbf{J}^{\eta\hat{}<\alpha>}, \mathbf{f}^{\eta\hat{}<\alpha>})$ be $(\bar M^{\eta,\rho_{u(\alpha)}}, \mathbf{J}^{\eta,\rho_{u(\alpha)}}, \mathbf{f}^{\eta,\rho_{u(\alpha)}})$.

Clearly they are as required. $\square_{8.14}$

* * *

8.18 Exercise: 1) Assume $\mathfrak{t}$ is an almost good λ-frame, $\mathfrak{u} = \mathfrak{u}^1_{\mathfrak{t}}$ from Definition 4.29 then for some $\mathfrak{u} - \{0,2\}$-appropriate $\mathfrak{h}$, for every $M \in K^{\mathfrak{t},\mathfrak{h}}_{\lambda^{++}}$ we have

(a) M is λ^+-saturated

(b) if $M_0 \in K_{\mathfrak{t}}$ is $\leq_{\mathfrak{K}[\mathfrak{t}]} M$ and $p \in \mathscr{S}^{\rm bs}_{\mathfrak{t}}(M_0)$

then $\dim(p,M) = \lambda^{++}$ that is, there is a sequence $\langle a_\alpha : \alpha < \lambda^{++}\rangle$ of members of M realizing p such that: if $M_0 \leq_{\mathfrak{t}} M_1 <_{\mathfrak{K}[\mathfrak{t}]} M$ then $\{\alpha < \lambda^{++} : \mathbf{tp}_{\mathfrak{t}}(a, M_1, M)$ does not fork over $M_0\}$ is a co-bounded subset of λ^{++}.

2) Similarly if $\mathfrak{t}$ has existence for $K^{3,\rm up}_{\mathfrak{s}}$ and $\mathfrak{u} = \mathfrak{u}^3_{\mathfrak{t}}$, see Definition 8.3.

8.19 Theorem. *$\dot{I}(\lambda^{++}, K^{\mathfrak{s}}) \geq \dot{I}(\lambda^{++}, K^{\mathfrak{s}}(\lambda^+\text{-saturated})) \geq \mu_{\text{unif}}(\lambda^{++}, 2^{\lambda^+})$ and even $\dot{I}(K^{\mathfrak{u},\mathfrak{h}}_{\lambda^{++}}) \geq \mu_{\text{unif}}(\lambda^{++}, 2^{\lambda^+})$ for any $\mathfrak{u}_{\mathfrak{s}} - \{0,2\}$-appropriate $\mathfrak{h}$ (so we can restrict ourselves to models λ^+-saturated above λ and if $\mathfrak{s} = \mathfrak{s}'$ also to $\tau_{\mathfrak{s}}$-fuller ones) <u>when</u>:*

- (a) $2^\lambda < 2^{\lambda^+} < 2^{\lambda^+}$
- (b) $\mathfrak{s}$ *has non-uniqueness for* WNF *(for every $M \in K_{\mathfrak{s}}$)*
- (c) K *is categorical in* λ
- (d) $\mathfrak{u}$ *has existence for* $K^{3,\text{up}}_{\mathfrak{s},\lambda^+}$.

Proof. We shall use 8.6, 8.12, 8.14. So assume toward contradiction that the conclusion fails. We try to apply Theorem 8.6, now its conclusion fails by our assumption toward contradiction, and clause (a) there which says "$2^\lambda < 2^{\lambda^+} < 2^{\lambda^{++}}$" holds by clause (a) of the present theorem. So necessarily clause (b) of Theorem 8.6 fails which means that $\mathfrak{u}$ has wnf-delayed uniqueness, see Definition 8.7.

Next we try to apply Theorem 8.14, again it assumption fails by our assumption toward contradiction, and among its assumptions clause (a) which says that "$2^\lambda < 2^{\lambda^+} < 2^{\lambda^{++}}$" holds by clause (a) of the present theorem, and clause (c) which says "$\mathfrak{s}$ has wnf-delayed uniqueness" has just been proved. So necessarily clause (b) of 8.14 fails which means that $\mathfrak{s}$ fails non-uniqueness for WNF, i.e. for some M.

Now we apply Observation 8.12, noting that its assumption "$\mathfrak{s}$ is categorical in λ" holds by clause (c) of the present theorem, so by the previous sentence one of the equivalent phrases the first fails, hence all of them. In particular $\mathfrak{s}$ has uniqueness for WNF. $\square_{8.19}$

§9 The combinatorial part

We deal here with the "relatively" pure-combinatorial parts. We do just what is necessary. We can get results on $\dot{I}\dot{E}(\partial^+, \mathfrak{K}_{\mathfrak{u}})$, we can weaken the cardinal arithmetic assumptions to $\emptyset \notin \text{DfWD}_\partial$, see [Sh:E45], we can weaken the demands on $\mathfrak{K}$; <u>but</u> not here.

Recall the obvious by the definitions:

9.1 Theorem. *If $2^\partial < 2^{\partial^+}$ <u>then</u> $\{M_\eta/\cong : \eta \in {}^{\partial^+}(2^\partial)$ and $\|M_\eta\| = \partial^+\}$ has cardinality $\geq \mu_{\text{unif}}(\partial^+, 2^\partial)$ <u>when</u> the following conditions hold:*

$\circledast$ (a) $\quad \bar{M} = \langle M_\eta : \eta \in {}^{\partial^+ >}(2^\partial)\rangle$

(*b*) *for* $\eta \in {}^{\partial^+ >}(2^\partial)$ *the model* M_η *has cardinality* $\le \partial$ *and for notational simplicity has universe an ordinal* $< \partial^+$

(*c*) $M_\eta \subseteq M_\nu$ *if* $\eta \triangleleft \nu \in {}^{\partial^+ >}(2^\partial)$, *so* <u>*no*</u> *a.e.c. appear here!*

(*d*) $\langle M_{\eta \restriction \alpha} : \alpha < \ell g(\eta)\rangle$ *is* $\subseteq$*-increasing continuous for any* $\eta \in {}^{\partial^+ >}(2^\partial)$

(*e*) $M_\eta := \cup\{M_{\eta\restriction\alpha} : \alpha < \partial^+\}$ *for* $\eta \in {}^{\partial^+}(2^\partial)$

(*f*) *if* $\eta \in {}^{\partial^+ >}(2^\partial)$ *and* $\alpha_1 < \alpha_2 < 2^\partial$ *and* $\eta\hat{\ }\langle\alpha_\ell\rangle \trianglelefteq \nu_\ell \in {}^\delta 2$ *for* $\ell = 1, 2$ *and* $\delta < \partial^+$ <u>*then*</u> M_{ν_2}, M_{ν_1} *are not isomorphic over* M_η *or just*

$(f)_1^-$ *for* $\eta \in {}^{\partial^+ >}2$, *there is* $\mathscr{U}_\eta \subseteq 2^\partial$ *of cardinality* 2^∂ *such that: if* $\alpha_0 \ne \alpha_1$ *are from* $\mathscr{U}_\eta$ *and* $\eta\hat{\ }\langle\alpha_\ell\rangle \trianglelefteq \nu_\ell \in {}^\delta 2$ *for* $\ell < 2$ *and* $\delta < \partial^+$ <u>*then*</u> M_{ν_0}, M_{ν_1} *are not isomorphic over* M_η.

Proof. Concerning clause $(\mathrm{f})_1^-$ we can by renaming get clause (f), so in the rest of the proof of 9.1 we can ignore clause $(\mathrm{f})_1^-$.

Note that $\Xi_0 := \{\eta \in {}^{\partial^+}(2^\partial) : \|M_\eta\| < \partial^+\}$ has cardinality $\le 2^\partial$ (because for each $\eta \in \Xi_0$ there is $\alpha_\eta < \partial^+$ such that $M_\eta = M_{\eta\restriction\alpha_\eta}$; and note that by clause (f) we have $\eta_1 \in \Xi_0 \wedge \eta_2 \in \Xi_0 \wedge \alpha_{\eta_1} = \alpha_{\eta_2} \wedge \eta_1 \restriction \alpha_{\eta_1} = \eta_2 \restriction \alpha_{\eta_2} \Rightarrow \eta_1 = \eta_2$). So by clause (b) of $\circledast$ it follows that $\eta \in {}^{\partial^+}(2^\partial)\backslash\Xi_0 \Rightarrow |M_\eta| = \partial^+$.

It suffices to assume that $\Xi \subseteq {}^{\partial^+}(2^\partial)$ has cardinality $< \mu_{\mathrm{unif}}(\partial^+, 2^\partial)$ and find $\eta \in {}^{\partial^+}(2^\partial)$ such that $\nu \in \Xi \Rightarrow M_\eta \not\cong M_\nu$, because without loss of generality $\Xi_0 \subseteq \Xi$.

Let $\langle \eta_\zeta : \zeta < |\Xi|\rangle$ list Ξ and let $N_\zeta := M_{\eta_\zeta}$ and toward contradiction for every $\nu \in {}^{\partial^+}(2^\partial)$ we can choose $\zeta_\nu = \zeta(\nu) < |\Xi|$ and an isomorphism f_ν from M_ν onto $N_{\zeta(\nu)}$, so f_ν is a function from M_ν onto $M_{\eta_{\zeta(\nu)}}$.

For $\zeta < |\Xi|$ let $W_\zeta = \{\nu \in {}^{\partial^+}(2^\partial) : \zeta_\nu = \zeta\}$, so clearly:

$(*)_1$ ${}^{\partial^+}(2^\partial)$ is equal to $\cup\{W_\zeta : \zeta < |\Xi|\}$.

[Why? Obvious by our assumption toward contradiction.]

$(*)_2$ if $i < \partial^+$ and $\rho \in {}^i(2^\partial)$, <u>then</u> there are no $\varepsilon_1 \ne \varepsilon_2 < 2^\partial$ such that $\rho\hat{\ }\langle\varepsilon_\ell\rangle \triangleleft \nu_\ell \in W_\zeta$ for $\ell = 1, 2$ and $f_{\nu_1} \restriction M_\eta = f_{\nu_2} \restriction M_\eta$.

[Why? By Clause (f) of the assumption.]

Together we get a contradiction to the definition of $\mu_{\mathrm{unif}}(\partial^+, 2^\partial)$, see Definition 0.4(7). $\square_{9.1}$

Similarly

9.2 Claim. *1) In 9.1 we can replace 2^∂ by $\langle \chi_i : i < \partial \rangle$ with $\chi_i \le 2^\partial$.*
2) Also we can weaken clause (f) or $(f)^-_1$ there by demanding $\delta = \partial^+$.
3) Assume $\mathfrak{K}$ is an a.e.c. and in 9.1 we demand $M_\nu \le_{\mathfrak{K}} M_\eta \in \mathfrak{K}$ for $\nu \triangleleft \eta \in {}^{\partial^+ >}(2^\partial)$. If we strengthen there clause $(f)^-_1$ by strengthening the conclusion to "if $\eta\hat{}\langle \ell \rangle \triangleleft \eta_\ell \in {}^{\partial^+}2$ for $\ell = 1,2$ then M_{ν_0}, M_{ν_1} cannot be $\le_{\mathfrak{K}}$-amalgamated over M_η" then:

- $(*)$ *for every $\Xi \subseteq {}^{\partial^+}(2^\partial)$ of cardinality $< \mu_{\rm unif}(\partial^+, 2^\partial)$ for some $\eta \in {}^{\partial^+}(2^\partial)$ the model M_η has cardinality ∂^+ and cannot be $\le_{\mathfrak{K}}$-embedded in M_ν for any $\nu \in \Xi$*
- $(**)$ *if $2^{\partial^+} > (2^\partial)^+$ then there is $\Xi \subseteq {}^{\partial^+}2$ of cardinality 2^{∂^+} such that if $\eta \ne \nu \in \Xi$ then M_η cannot be $\le_{\mathfrak{K}}$-embedded into M_ν.*

Proof. Left to the reader (easier than 9.7 below and will not be used here). $\square_{9.2}$

Remark. Why do we prefer to state 9.1? As this is how it is used.

9.3 Lemma. *Assuming $2^\theta = 2^{<\partial} < 2^\partial$ (and naturally but not used $2^\partial < 2^{\partial^+}$) and $\circledast(a)-(e)$ of 9.1, a sufficient condition for clause $(f)^-_1$ of 9.1 is:*

- $(a)^+$ *$\bar{M} = \langle M_\eta : \eta \in {}^{\partial^+ >}(2^\partial) \rangle$ and $\langle M^*_{\eta,\zeta} : \zeta < \partial \rangle$ is $\subseteq$-increasing with union M_η such that $\zeta < \partial \Rightarrow \|M_{\eta,\zeta}\| < \partial$*
- $(f)^-_2$ *for each $\eta \in {}^{\partial^+ >}2$ we can find $\langle M_{\eta,\rho} : \rho \in {}^{\partial \ge}2 \rangle$ such that*
  - (α) *$\langle M_{\eta,\rho} : \rho \in {}^\partial 2 \rangle$ is a subsequence of $\langle M_{\eta\hat{}<\alpha>} : \alpha < 2^\partial \rangle$ with no repetitions so $M_{\eta,\rho} = M_{\eta\hat{}<\alpha(\rho)>}$ for some one-to-one function $\rho \mapsto \alpha(\rho)$ from ${}^\partial 2$ to 2^∂*
  - (β) *if $\rho \in {}^{\partial >}2$ then $M_{\eta,\rho} \in K_{<\partial}$*
  - (γ) *if $\rho \in {}^{\partial >}2$ then $\langle M_{\eta,\rho \restriction \alpha} : \alpha \le \ell g(\rho) \rangle$ is $\subseteq$-increasing continuous*
  - (δ) *$\cup \{M_{\eta,\rho \restriction \varepsilon} : \varepsilon < \partial\}$ is equal to $M_{\eta,\rho} = M_{\eta\hat{}<\alpha(\rho)>}$ for any $\rho \in {}^\partial 2$*
  - (ε) *∂ is regular uncountable and for some sequence $\langle S_\varepsilon : \varepsilon < \partial \rangle$ of pairwise disjoint non-small stationary subsets of ∂ (i.e. $\varepsilon < \partial \Rightarrow S_\varepsilon \in ({\rm WDmId}_\partial)^+$) we have*
    - $(*)$ *for every $\varepsilon < \partial$, there is a pair $(\bar{g}, \mathbf{c}) = (\bar{g}^\varepsilon, \mathbf{c}^\varepsilon)$, may not depend on ε such that:*
      - $\bullet_1$ *$\bar{g} = \langle g_{\eta,\rho} : \rho \in {}^\partial 2 \rangle$*
      - $\bullet_2$ *$g_{\eta,\rho}$ is a function from ∂ to $\mathscr{H}_{<\partial}(\partial^+)$*
      - $\bullet_3$ *if $2^\partial > \partial^+$ and $\rho_0, \rho_1 \in {}^\partial 2, \rho_1 \restriction S_\varepsilon$ is constantly zero,*

$\delta < \partial^+, \eta\hat{\ }\langle\alpha(\rho_\ell)\rangle \trianglelefteq \nu_\ell \in {}^\delta(2^\partial)$ for $\ell = 0,1$ and f is an isomorphism from M_{ν_0} onto M_{ν_1} <u>then</u> for some club E of ∂, if $\zeta \in E \cap S_\varepsilon$ we have
$\rho_0(\zeta) = \mathbf{c}^\varepsilon(\rho_0 \restriction \zeta, M_{\eta,\rho_0\restriction\zeta}, \rho_1 \restriction \zeta, M_{\eta,\rho_1\restriction\zeta}, g_{\eta,\rho_0} \restriction \zeta,$
$g_{\eta,\rho_1} \restriction \zeta, M_{\nu_0,\zeta}, M_{\nu_\zeta}, f \restriction M_{\eta,\rho_0\restriction\zeta})$

$\bullet_4$ *if $2^\partial = \partial^+$: as above but $\mathbf{c}$ is preserved by any partial order preserving function from ∂^+ to ∂^+ extending $\mathrm{id}_{M^\eta_\partial}$.*

Remark. 1) We can immitate 9.7.
2) If $2^\partial = \partial^+$ then it follows that $\partial = \partial^{<\partial}$, so they give us stronger ways to construct.

Proof. First

$\boxtimes$ for $\eta \in {}^{\partial^+>}(2^\partial)$ and $\varepsilon < \partial$ there is $\varrho_{\eta,\varepsilon} \in {}^{(S_\varepsilon)}2$ such that:

(∗) if $\rho_0 \neq \rho_1$ are from ${}^\partial 2, \eta\hat{\ }\langle\alpha(\rho_\ell)\rangle \triangleleft \nu_\ell \in {}^\delta 2, \delta < \partial^+$ and f is an isomorphism from M_{ν_0} onto M_{ν_1} <u>then</u> for stationary many $\zeta \in S_\varepsilon$ we have:

$$\begin{aligned}\varrho_\eta(\zeta) = &\mathbf{c}^\varepsilon(\rho_0 \restriction \zeta, M_{\eta,\rho_0\restriction\zeta}, \rho_1 \restriction \zeta, M_{\eta,\rho_1\restriction\zeta}, g_{\eta,\rho_0} \restriction \zeta, g_{\eta,\rho_1} \restriction \zeta,\\ &M^*_{\nu_0,\zeta}, M^*_{\nu_1,\zeta}, f \restriction M_{\eta,\rho_0\restriction\zeta})\end{aligned}$$

[Why? First if $2^\theta \geq \partial^+$, use the definition of $S_\varepsilon \notin \mathrm{WDmId}(\partial)$, (see more in the proof of 9.6). If $2^\theta = \partial \wedge 2^\partial = \partial^+$, the proof is similar using the invariance of $\mathbf{c}^\varepsilon$, i.e. $\bullet_4$.

Lastly, if $2^\theta = \partial \wedge 2^\partial > \partial^+$, use $\mu_{\mathrm{wd}}(\partial) > \partial^+$, see 0.5(1A).]

Let $\eta \in {}^{\partial^+>}(2^\partial)$. For any $w \subseteq \partial$ we define $\rho_{\eta,w} \in {}^\partial 2$ as follows: $\rho_{\eta,w}(i)$ is $\varrho_{\eta,\varepsilon}$ <u>if</u> for some $\varepsilon < \partial$ and $\ell < 2$ we have $i \in S_{2\varepsilon+\ell} \wedge [\varepsilon \in w \equiv \ell = 1]$ and is zero otherwise. So $\{\alpha_\eta(\rho_{\eta,w}) : w \subset \partial\}$ is as required in $(\mathrm{f})^-_1$. $\square_{9.3}$

9.4 Theorem. *If $2^\partial < 2^{\partial^+}$ and $\mu = \mu_{\rm unif}(\partial^+, 2^\partial)$ <u>then</u>:*

(A) $\emptyset \notin \mathrm{UnfTId}_\mu(\partial^+)$

(B) $\mathrm{UnfTId}_{\mu_1}(\partial^+)$ *is μ_1-complete when $\aleph_0 \le \mu_1 = \mathrm{cf}(\mu_1) < \mu$; see 0.4(4),(5)*

(C) $\mu = 2^{\partial^+}$ *except maybe <u>when</u> (all the conditions below hold):*

$\circledast$ (a) $\mu < \beth_\omega$

(b) $\mu^{\aleph_0} = 2^{\partial^+}$

(c) *there is a family $\mathscr{A} \subseteq [\mu]^{\partial^+}$ of cardinality $\ge 2^{\partial^+}$ such that the intersection of any two distinct members of $\mathscr{A}$ is finite.*

Remark. So in the aleph sequence μ is much larger than 2^∂, when $\mu \ne 2^{\partial^+}$.

Proof. By [Sh:f, AP,1.16] we have clauses (b) + (c) of $\circledast$ and they imply clause (a) by [Sh 460] (or see [Sh 829]). $\square_{9.4}$

9.5 Claim. *Assume $\partial > \theta \ge \aleph_0$ is regular and $2^\theta = 2^{<\partial} < 2^\partial$. <u>Then</u> $\{M_\eta/\cong: \eta \in {}^\partial 2$ and M_η has cardinality $\partial\}$ has cardinality 2^∂ <u>when</u> the following condition holds:*

$\circledast$ (a) $\bar M = \langle M_\eta : \eta \in {}^{\partial \ge}2\rangle$ *with M_η a τ-model*

(b) *for $\eta \in {}^\partial 2$, $\langle M_{\eta \restriction \alpha} : \alpha \le \partial\rangle$ is $\subseteq$-increasing continuous*

(c) *if $\eta \in {}^{\partial >}2$ and $\eta\hat{}\langle \ell\rangle \trianglelefteq \nu_\ell \in {}^\alpha 2$ for $\ell = 0,1$ and $\alpha < \partial$ <u>then</u> M_{ν_0}, M_{ν_1} are not isomorphic over $M_{<>}$ or just for $\alpha = \partial$*

(d) *$M_{<>}$ has cardinality $< \partial$*

(e) *M_η has cardinality $\le \partial$ for $\eta \in {}^{\partial >}2$.*

Proof. As in the proof of 9.1 we can ignore the $\eta \in {}^\partial 2$ for which M_η has cardinality $< \partial$.

As $\partial^{\|M_{<>}\|} \le 2^{<\partial} < 2^\partial$ this is obvious, see I.88r-0.9 or see Case 1 in the proof of 9.7 below. $\square_{9.5}$

The following is used in VI.E46-3c.22 (and can be used in VI.E46-2b.32; but compare with 9.5!)

9.6 Claim. *The set $\{M_\eta/\cong: \eta \in {}^\partial 2$ and M_η has cardinality $\partial\}$ has cardinality $\geq \mu$ when*

$\boxtimes_1$ $\partial = \mathrm{cf}(\partial) > \aleph_0$ *and*

- *(a)* M_η *is a* τ*-model of cardinality* $< \partial$ *for* $\eta \in {}^{\partial >}2$
- *(b)* *for each* $\eta \in {}^\partial 2$, $\langle M_{\eta \restriction \alpha} : \alpha < \partial\rangle$ *is* $\subseteq$*-increasing continuous with union, called* M_η
- *(c)* *if* $\eta \in {}^{\partial >}2, \eta\hat{\ }\langle \ell\rangle \triangleleft \rho_\ell \in {}^\partial 2$ *for* $\ell = 1, 2$ *<u>then</u>* M_{ρ_1}, M_{ρ_2} *are not isomorphic over* M_η

$\boxtimes_2$ $\partial \notin \mathrm{WDmId}_{<\mu}(\partial)$, *e.g.* $\mu = \mu_{\mathrm{wd}}(\partial)$.

Proof. Let $\Xi = \{\eta \in {}^\partial 2 : M_\eta$ has cardinality $< \partial\}$ and for $\eta \in \Xi$ let $\alpha_\eta = \min\{\alpha \leq \partial : M_\eta = M_{\eta \restriction \alpha}\}$, clearly

$\boxdot_1$ (a) $\eta \in \Xi$ implies $\alpha_\eta < \partial$

(b) if $\eta \in \Xi$ and $\eta \restriction \alpha \triangleleft \nu \in \Xi \backslash \{\eta\}$ then $\alpha_\nu > \alpha_\eta$.

For each $\varrho \in {}^\partial 2$ we define $F_\varrho : {}^{\partial \geq}2 \to {}^{\partial \geq}2$ by: for $\eta \in {}^\alpha 2$ let $F_\varrho(\eta) \in {}^{2\ell g(\eta)}2$ be defined by $(F_\varrho(\eta))(2i) = \eta(i), (F_\varrho(\eta))(2i+1) = \varrho(i)$ for $i < \alpha$. Easily $\langle \mathrm{Rang}(F_\varrho) : \varrho \in {}^\partial 2\rangle$ are pairwise disjoint, hence for some $\varrho \in {}^\gamma 2$, the sets $\mathrm{Rang}(F_\varrho)$ is disjoint to Ξ so without loss of generality (by renaming):

$\boxdot_2$ $\Xi = \emptyset$.

Let $\{N_\varepsilon : \varepsilon < \varepsilon_*\}$ be a maximal subset of $\{M_\rho : \rho \in {}^\partial 2\}$ consisting of pairwise non-isomorphic models.

Without loss of generality the universe of each $M_\eta, \eta \in {}^{\partial >}2$ is an ordinal $\gamma_\eta < \partial$ and so the universe of each $M_\eta, \eta \in {}^\partial 2$ is $\gamma_\eta := \cup\{\gamma_{\eta \restriction i} : i < \partial\} = \partial$, in particular the universe of N_ε is ∂ and $\eta \in {}^\partial 2 \Rightarrow \gamma_\eta = \partial$. For $\alpha < \partial$ and $\eta \in {}^\alpha 2$ let the function h_η be $h_\eta(i) = M_{\eta \restriction (i+1)}$ for $i < \ell g(\eta)$. For each $\varepsilon < \varepsilon_*$ we define $\Xi_\varepsilon \subseteq {}^\partial 2$ by $\Xi_\varepsilon = \{\eta \in {}^\partial 2 : M_\eta$ is isomorphic to $N_\varepsilon\}$.

For $\eta \in \Xi_\varepsilon$ choose $f^\varepsilon_\eta : M_\eta \to N_\varepsilon$, an isomorphism, hence $f^\varepsilon_\eta \in {}^\partial\partial$.

By the assumption

$\boxdot_3$ if $\varepsilon < \varepsilon_*$ and $\eta \in {}^{\partial >}2$ and $\eta\hat{\ }\langle \ell\rangle \triangleleft \nu_\ell \in \Xi_\varepsilon$ for $\ell = 0, 1$ then $f^\varepsilon_{\nu_0} \restriction \gamma_\eta \neq f^\varepsilon_{\nu_1} \restriction \gamma_\eta$.

We also for each $\varepsilon < \varepsilon_*$ define a function (= colouring) $\mathbf{c}_\varepsilon$ from $\bigcup\limits_{\alpha < \partial} ({}^\alpha 2 \times {}^\alpha \partial)$ to $\{0, 1\}$ by:

$\boxdot_4$ $\mathbf{c}_\varepsilon(\eta, f)$ is : 0 <u>if</u> there is ν such that $\eta \triangleleft \nu \in \Xi_\varepsilon$ and $f \subseteq f^\varepsilon_\nu$ and $\nu(\ell g(\eta)) = 0$
$\mathbf{c}_\varepsilon(\eta, f)$ is: 1 <u>if</u> otherwise.

Now for any $\eta \in \Xi_\varepsilon$, the set

$$E^\varepsilon_\eta = \{\delta < \partial : \gamma_{\eta \restriction \delta} = \delta \text{ and } f^\varepsilon_\eta \restriction \delta \text{ is a function from } \delta \text{ to } \delta\}$$

is clearly a club of ∂.

Now

$\boxdot_5$ if $\varepsilon < \varepsilon_*, \eta \in \Xi_\varepsilon$ and $\delta \in E^\varepsilon_\eta$ then $\mathbf{c}_\varepsilon(\eta \restriction \delta, f^\varepsilon_\eta \restriction \delta) = \eta(\delta)$.

[Why? If $\eta(\delta) = 0$ then $\eta \restriction \delta$ witness that $\mathbf{c}_\varepsilon(\eta \restriction \delta, f^\varepsilon_\eta \restriction \delta) = 0$. If $\eta(\delta) = 1$ just recall $\boxdot_3$.]

Hence we have $\Xi_\varepsilon \in$ WDmTId(∂). To get a contradiction it is enough to prove $\cup\{\Xi_\varepsilon : \varepsilon < \varepsilon_*\} \neq {}^\partial 2$, but as $\varepsilon_* < \mu$ clearly $\bigcup\limits_{\varepsilon < \varepsilon_*} \Xi_\varepsilon$ belongs to WDmId$_{<\mu}(\partial)$ hence is not ${}^\partial 2$, so we are done. $\square_{9.6}$

The following is used in VI. E46-2b.32, VI. E46-3c.28, VI. E46-3c.22 which repeat the division to cases.

9.7 Claim. *The set $\{M_\eta / \cong : \eta \in {}^\partial 2$ and $\|M_\eta\| = \partial\}$ has cardinality 2^∂ <u>when</u>:*

$\boxtimes_1$ *M_η is a τ-model of cardinality $< \partial$ for $\eta \in {}^{\partial >}2, \langle M_{\eta \restriction \alpha} : \alpha \leq \ell g(\eta)\rangle$ is $\subseteq$-increasing continuous, and: if $\delta < \delta(1) < \partial$ are limit ordinals, $\eta_0, \eta_1 \in {}^\delta 2$ and $\eta_0{}^\frown\langle \ell \rangle \triangleleft \nu_\ell \in {}^{\delta(1)}2$ and $\eta_1{}^\frown\langle 0 \rangle \triangleleft \nu'_\ell \in {}^{\delta(1)}2$ for $\ell = 0, 1$ <u>then</u> there are no f_0, f_1 such that*

- (α) *f_ℓ is an isomorphism from M_{ν_ℓ} onto $M_{\nu'_\ell}$ for $\ell = 0, 1$*
- (β) *$f_0 \restriction M_{\eta_0} = f_1 \restriction M_{\eta_0}$ and $M_{\eta_1} = f_0(M_{\eta_0})$*
- (γ) *for some $\rho_0, \rho_1 \in {}^\partial 2$ we have $\nu'_\ell \triangleleft \rho_\ell$ for $\ell = 0, 1$ and M_{ρ_0}, M_{ρ_1} are isomorphic over M_{η_1}*

$\boxtimes_2$ *$\partial = \mathrm{cf}(\partial) > \aleph_0$, $\partial \notin$ WDmId$_{<\mu}(\partial)$ (hence $2^{<\partial} < 2^\partial$) and moreover*

$\boxtimes_3$ *∂ is a successor cardinal, or at least there is no ∂-saturated normal ideal on ∂, or at least WDmId(∂) is not ∂-saturated (which holds if for some $\theta < \partial, \{\delta < \partial : \mathrm{cf}(\delta) = \theta\} \notin$ WDmId(∂) because the ideal is normal).*

9.8 Remark. 1) Compare with I. 88r-3.5 - which is quite closed but speak on $\mathfrak{K}$ rather than on a specific $\langle M_\eta : \eta \in {}^{\partial >}2\rangle$. Can we get $\dot{I}\dot{E}_{\mathfrak{K}}(\partial, \mathfrak{K}) = 2^\partial$?

Also λ^+ there corresponds to ∂ here, a minor change.
2) The parallel claim was inaccurate in the [Sh 576, §3].
3) Used in VI. E46-2b.32.

Proof of 9.7. Easily, as in the proof of 9.6 without loss of generality

$\boxdot_1$ $\eta \in {}^\partial 2 \Rightarrow \|M_\eta\| = \partial$ while, of course, preserving $\boxtimes_1$.

We divide the proof into cases according to the answer to the following:

Question: Is there $\eta^* \in {}^{\partial>}2$ such that for every ν satisfying $\eta^* \trianglelefteq \nu \in {}^{\partial>}2$ there are $\rho_0, \rho_1 \in {}^{\partial>}2$ such that: $\nu \triangleleft \rho_0, \nu \trianglelefteq \rho_1$, and for any $\nu_0, \nu_1 \in {}^\partial 2$ satisfying $\rho_\ell \triangleleft \nu_\ell$, (for $\ell = 0, 1$) the models M_{ν_0}, M_{ν_1} are not isomorphic over M_{η^*}?

But first we can find a function $h : {}^{\partial>}2 \to {}^{\partial>}2$, such that:

- $(*)$ the function h is one-to-one, mapping ${}^{\partial>}2$ to ${}^{\partial>}2$, preserving $\triangleleft$, satisfying $(h(\nu))\hat{}\langle\ell\rangle \trianglelefteq h(\nu\hat{}\langle\ell\rangle)$ and h is continuous, for $\nu \in {}^\partial 2$ we let $h(\nu) := \bigcup\limits_{\alpha<\partial} h(\nu \restriction \alpha)$, so $\ell g(\eta) < \partial \Leftrightarrow \ell g(h(\eta)) < \partial$ and:
- $(b)_{\rm yes}$ when the answer to the question above is yes, it is exemplified by $\eta^* = h(\langle\rangle)$ and $M_{h(\rho_0)}, M_{h(\rho_1)}$ are not isomorphic over $M_{h(<>)}$ whenever $\nu \in {}^{\partial>}2$ and $h(\nu\hat{} < \ell >) \triangleleft \rho_\ell \in {}^\partial 2$ for $\ell = 0, 1$
- $(b)_{\rm no}$ when the answer to the question above is no, $h(\langle\rangle) = \langle\rangle$ and if $\alpha + 1 < \beta < \partial, \eta \in {}^{\alpha+1}2$ and $h(\eta) \triangleleft \rho_\ell \in {}^\beta 2$ for $\ell = 1, 2$ then we can find ν_1, ν_2 and g^* such that $\rho_\ell \triangleleft \nu_\ell \in {}^\partial 2$ and g^* is an isomorphism from M_{ν_1} onto M_{ν_2} over $M_{h(\eta \restriction \alpha)}$.

[Why can we get $(b)_{\rm no}$? We choose $h(\eta)$ for $\eta \in {}^\alpha 2$ by induction on α such that $h(\eta) = \eta$ for $\alpha = 0, h(\eta) = \cup\{h(\eta \restriction \beta) : \beta < \alpha\}$ when α is a limit ordinal, and if $\alpha = \beta + 1, \ell < 2$ apply the assumption ("the answer is no") with $h(\eta)\hat{} < \ell >$ standing for η^* and let $h(\eta\hat{} < \ell >)$ be a counterexample to "for every ν"; so we get even more then the promise; the isomorphism is over $M_{h(\eta\restriction\alpha)\hat{}\langle\ell\rangle}$ rather than $M_{h(\eta\restriction\alpha)}$, and note that $h(\eta)\hat{}\langle\ell\rangle \triangleleft h(\eta\hat{}\langle\ell\rangle)$.]

Case 1: The answer is yes.

We do not use the non-∂-saturation of WDmId(∂) in this case. Without loss of generality h is the identity, by renaming.
For any $\eta \in {}^\partial 2$ and $\subseteq$-embedding g of $M_{\langle\rangle}$ into $M_\eta := \bigcup\limits_{\alpha<\partial} M_{\eta\restriction\alpha}$, let

$$\Xi_{\eta,g} := \{\nu \in {}^\partial 2 : \text{there is an isomorphism from } M_\nu \text{ onto } M_\eta \text{ extending } g\}$$

$$\Xi_\eta := \{\nu \in {}^\partial 2 : \text{there is an isomorphism from } M_\nu \text{ onto } M_\eta\}.$$

So:

$\boxdot_2$ $|\Xi_{\eta,g}| \le 1$ for any g and $\eta \in {}^\partial 2$.

[Why? As if $\nu_0, \nu_1 \in \Xi_{\eta,g}$ are distinct then for some ordinal $\alpha < \partial$ and $\nu \in {}^\alpha 2$ we have $\nu := \nu_0 \restriction \alpha = \nu_1 \restriction \alpha, \nu_0(\alpha) \ne \nu_1(\alpha)$ and use the choice of $h(\nu\hat{}\langle \ell \rangle))$, see $(b)_{\text{yes}}$ above.]
Since $\Xi_\eta = \cup\{\Xi_{\eta,g} : g \text{ is a } \le_{\mathfrak{K}} \text{-embedding of } M_{\langle\rangle} \text{ into } M_\eta\}$, we have

$\boxdot_3$ $|\Xi_\eta| \le \partial^{\|M_{\eta^*}\|} \le 2^{<\partial}$.

Hence we can by induction on $\zeta < 2^\partial$ choose $\eta_\zeta \in {}^\partial 2 \backslash \bigcup\limits_{\xi<\zeta} \Xi_{\eta_\xi}$, (exist by cardinality considerations as $2^{<\partial} < 2^\partial$). Then $\xi < \zeta \Rightarrow M_{\eta_\xi} \ncong M_{\eta_\zeta}$ so we have proved the desired conclusion.

<u>Case 2</u>: The answer is no.
Without loss of generality M_η has as universe the ordinal $\gamma_\eta < \partial$ for $\eta \in {}^{\partial>}2$. Let $\langle S_i : i < \partial \rangle$ be a partition of ∂ to sets, none of which is in $\mathrm{WDmId}(\partial)$, possible by the assumption $\boxtimes_3$. For each $i < \partial$ we define a function $\mathbf{c}_i$ as follows:

$\boxdot_4$ if $\delta \in S_i$ and $\eta, \nu \in {}^\delta 2$ and $\gamma_\eta = \gamma_\nu = \delta = \gamma_{h(\eta)} = \gamma_{h(\nu)}$, and $f : \delta \to \delta$ then
- (a) $\mathbf{c}_i(\eta, \nu, f) = 0$ <u>if</u> we can find $\eta_1, \eta_2 \in {}^\partial 2$ satisfying $h(\eta)\hat{}\langle 0 \rangle \triangleleft \eta_1$ and $h(\nu)\hat{}\langle 0 \rangle \triangleleft \eta_2$ such that f can be extended to an isomoprhism from M_{η_1} onto M_{η_2}
- (b) $\mathbf{c}_i(\eta, \nu, f) = 1$ otherwise.

So for $i < \partial$, as $S_i \notin \mathrm{WDmId}(\partial)$, for some $\varrho_i^* \in {}^\partial 2$ we have:

$(*)_i$ for every $\eta \in {}^\partial 2, \nu \in {}^\partial 2$ and $f \in {}^\partial \partial$ the following set of ordinals is stationary:

$$\{\delta \in S_i : \mathbf{c}_i(\eta \restriction \delta, \nu \restriction \delta, f \restriction \delta) = \varrho_i^*(\delta)\}.$$

Now for any $X \subseteq \partial$ let $\eta_X \in {}^\partial 2$ be defined by:

$\boxdot_5$ if $\alpha \in S_i$ then $i \in X \Rightarrow \eta_X(\alpha) = 1 - \varrho_i^*(\alpha)$ and $i \notin X \Rightarrow \eta_X(\alpha) = 0$.

For $X \subseteq \partial$ let $\rho_X := \eta_{\{2i:i\in X\}\cup\{2i+1:i\in\partial\backslash X\}} \in {}^\partial 2$. Now we shall show

$\oplus$ if $X, Y \subseteq \partial$, and $X \ne Y$ then $M_{h(\rho_X)}$ is not isomorphic to $M_{h(\rho_Y)}$.

Clearly $\oplus$ will suffice for finishing the proof.

Assume toward a contradiction that f is an isomorphism of $M_{h(\rho_X)}$ onto $M_{h(\rho_Y)}$; as $X \neq Y$ there is i such that $i \in X \Leftrightarrow i \notin Y$ so there is $j \in \{2i, 2i+1\}$ such that

$\boxdot_6$ $\rho_X \restriction S_j = \langle 1 - \varrho^*_j(\alpha) : \alpha \in S_j \rangle$ and $\rho_Y \restriction S_j$ is identically zero.

Clearly the set $E = \{\delta : f$ maps δ onto δ and $h(\rho_X \restriction \delta), h(\rho_Y \restriction \delta) \in {}^{\delta}2$ and the universes of $M_{h(\rho_X \restriction \delta)}, M_{h(\rho_Y \restriction \delta)}$ are $\delta\}$ is a club of ∂ and hence $S_j \cap E \neq \emptyset$.

So if $\delta \in S_j \cap E$ then f extends $f \restriction M_{h(\rho_X) \restriction \delta}$ and f is an isomorphism from $M_{h(\rho_X)}$ onto $M_{h(\rho_Y)}$; by the choice of ϱ^*_j we can choose $\delta \in S_j \cap E$ such that:

$\boxdot_7$ $\mathbf{c}_j(\rho_X \restriction \delta, \rho_Y \restriction \delta, f \restriction \delta) = \varrho^*_j(\delta)$.

Also by the choice of j, i.e. $\boxdot_6$ we have

$\boxdot_8$ $\rho_X(\delta) = 1 - \varrho^*_i(\delta)$ and $\rho_Y(\delta) = 0$.

<u>Subcase 2A</u>: $\rho_X(\delta) = 0$.

Now $\rho_X \restriction \delta \triangleleft (\rho_X \restriction \delta)\hat{\ }\langle \rho_X(\delta)\rangle = (\rho_X \restriction \delta)\hat{\ } < 0 > \triangleleft \rho_X \in {}^{\partial}2$ and $(\rho_Y \restriction \delta) \triangleleft (\rho_Y \restriction \delta)\hat{\ }\langle \rho_Y(\delta)\rangle = \rho_Y\hat{\ } < 0 > \triangleleft \rho_Y \in {}^{\partial}2$ (as $\rho_X(\delta) = 0$ by the case and $\rho_Y(\delta) = 0$ as $\delta \in S_j$ and the choice of j, i.e. by $\boxdot_6$). Hence f, ρ_X, ρ_Y witness that by the definition of $\mathbf{c}_j$ we get

$\otimes_1$ $\mathbf{c}_j(\rho_X \restriction \delta, \rho_Y \restriction \delta, f \restriction \delta) = 0$.

Also, by $\boxdot_8$

$\otimes_2$ $0 = \rho_X(\delta) = 1 - \varrho^*_j(\delta)$ so $\varrho^*_j(\delta) = 1$.

But $\otimes_1 + \otimes_2$ contradict the choice of δ, (indirectly the choice of ϱ^*_j), i.e., contradicts $\boxdot_7$.

<u>Subcase 2B</u>: $\rho_X(\delta) = 1$.

By $\boxdot_7$ and $\boxdot_6$ and the case assumption we have $\mathbf{c}_j(\rho_X \restriction \delta, \rho_Y \restriction \delta, f \restriction \delta) = \varrho^*_j(\delta) = 1 - \rho_X(\delta) = 0$ hence by the definition of $\mathbf{c}_j$ there are $\eta_1, \eta_2 \in {}^{\partial}2$ such that $h(\rho_X \restriction \delta)\hat{\ } < 0 > \triangleleft \eta_1, h(\rho_Y \restriction \delta)\hat{\ } < 0 > \triangleleft \eta_2$, and there is an isomorphism g from M_{η_1} onto M_{η_2} extending $f \restriction \delta$. There is $\delta_1 \in (\delta, \partial)$ such that: f maps $M_{h(\rho_X) \restriction \delta_1}$ onto $M_{h(\rho_Y) \restriction \delta_1}$ and g maps $M_{\eta_1 \restriction \delta_1}$ onto $M_{\eta_2 \restriction \delta_2}$. Now by the choice of h, i.e., clause $(b)_{\rm no}$ above, with $h(\rho_Y \restriction \delta)\hat{\ } < 0 >, \eta_2 \restriction \delta_1, h(\rho_Y) \restriction \delta_1$ here standing for ν, ρ_1, ρ_2 there and get ν_1, ν_2, g^* as there so $\eta_2 \restriction \delta_1 \triangleleft \nu_1 \in {}^{\partial}2, h(\rho_Y) \restriction \delta_1 \triangleleft \nu_2 \in {}^{\partial}2$ and g^* is an isomorphism form M_{ν_1} onto M_{ν_2} over $M_{h(\rho_Y) \restriction \delta\hat{\ }<0>}$. So this contradicts $\boxtimes_1$ in the assumption of the claim with $\delta, \delta_1, h(\rho_X \restriction \delta), h(\rho_Y \restriction \delta), \eta_1 \restriction \delta_1, h(\rho_X \restriction \delta_1), \eta_2 \restriction \delta_1, h(\rho_Y) \restriction \delta_1, f \restriction M_{\eta_1 \restriction \delta_1}, g \restriction M_{h(\rho_X \restriction \delta_1)}, \nu_1, \nu_2$ here standing for $\delta, \delta(1), \eta_0, \eta_1, \nu_0, \nu_1, \nu'_0, \nu'_1, f_0, f_1, \rho_0, \rho_1$ there. $\square_{9.7}$

§10 Proof of the non-structure theorems with choice functions

When we try to apply several of the coding properties, we have to use the weak diamond (as e.g. in 9.7), but in order to use it we have to fix some quite arbitrary choices; this is the role of the $\bar{\mathbb{F}}$'s here. Of course, we can weaken 10.1, but no need here.

10.1 Hypothesis. $\mathfrak{u}$ is a nice ∂-construction framework (so ∂ is regular uncountable) and τ is a $\mathfrak{u}$-sub-vocabulary.

10.2 Definition. We call a model $M \in \mathfrak{K}_{\mathfrak{u}}$ standard if $M \in K^\circ_{\mathfrak{u}} := \{M \in K_{\mathfrak{u}}$: every member of M is an ordinal $< \partial^+\}$ and $\mathfrak{K}^\circ_{\mathfrak{u}} = (K^\circ_{\mathfrak{u}}, \leq_{\mathfrak{K}} \restriction K^o_{\mathfrak{u}})$.

Convention: Models will be standard in this section if not said otherwise.

10.3 Definition. 1) Let $K^{\rm rt}_\partial = K^{\rm rt}_{\mathfrak{u}}$ be the class of quadruples $(\bar{M}, \bar{\mathbf{J}}, \mathbf{f}, \bar{\mathbb{F}})$ such that:

- (A) $(\bar{M}, \bar{\mathbf{J}}, \mathbf{f}) \in K^{\rm qt}_{\mathfrak{u}}$ recalling 1.15, and $M_\partial = \cup\{M_\alpha : \alpha < \partial\}$ has universe some ordinal $< \partial^+$ divisible by ∂ hence M_α is standard for $\alpha < \partial$
- (B) $\bar{\mathbb{F}} = \langle \mathbb{F}_\alpha : \alpha < \partial\rangle$ where $\mathbb{F}_\alpha$ is a $\mathfrak{u}$-amalgamation choice function, see part (2) below and[32] if $2^\partial = \partial^+$ then each $\mathbb{F}_\alpha$ has strong uniqueness, see Definition 10.4(2) below.

2) We say that $\mathbb{F}$ is a $\mathfrak{u}$-amalgamation function when:

- (a) $\mathrm{Dom}(\mathbb{F}) \subseteq \{(M_0, M_1, M_2, \mathbf{J}_1, \mathbf{J}_2, A) : M_\ell \in \mathfrak{K}^\circ_{\mathfrak{u}}$ for $\ell \leq 2, M_0 \leq_{\mathfrak{K}} M_\ell, (M_0, M_\ell, \mathbf{J}_\ell) \in \mathrm{FR}_\ell$ for $\ell = 1,2$ and $M_1 \cap M_2 = M_0$ and $M_1 \cup M_2 \subseteq A \subseteq \partial^+$, and $|A \backslash M_1 \backslash M_2| < \partial\}$
- (b) if $\mathbb{F}(M_0, M_1, M_2, \mathbf{J}_1, \mathbf{J}_2, A)$ is well defined then it has the form $(M_3, \mathbf{J}^+_1, \mathbf{J}^+_2)$ such that
  - (α) $M_\ell \leq_{\mathfrak{K}} M_3 \in K^\circ_{<\partial}$ for $\ell = 1,2$
  - (β) $|M_3| = A$
  - (γ) $(M_0, M_\ell, \mathbf{J}_\ell) \leq^\ell_{\mathfrak{u}} (M_{3-\ell}, M_3, \mathbf{J}^+_\ell)$ for $\ell = 1,2$.
- (c) if $(M_0, M_1, M_2, \mathbf{J}_1, \mathbf{J}_2)$ are as in clause (a) then for some A we have: $\mathbb{F}(M_0, M_1, M_2, \mathbf{J}_1, \mathbf{J}_2, A)$ is well defined and for any such A, the set $A \backslash M_1 \backslash M_2$ is disjoint to $\sup\{\gamma + 1 : \gamma \in M_1$ or $\gamma \in M_2\}$

[32] can demand this always

(*d*) if[33] $\mathbb{F}(M_0, M_1, M_2, \mathbf{J}_1, \mathbf{J}_2, A^k)$ is well defined for $k = 1, 2$ then $|A^1 \backslash M_1 \backslash M_2| = |A^2 \backslash M_1 \backslash M_2|$
moreover

(*e*) if $\mathbb{F}(M_0, M_1, M_2, \mathbf{J}_1, \mathbf{J}_2, A^1)$ is well defined and $M_1 \cup M_2 \subseteq A^2 \subseteq \partial^+$ and $\mathrm{otp}(A^2 \backslash M_1 \backslash M_2) = \mathrm{otp}(A^1 \backslash M_1 \backslash M_2)$ then also $\mathbb{F}(M_0, M_1, M_2, \mathbf{J}_1, \mathbf{J}_2, A^2)$ is well defined.

3) Let $(\bar{M}^1, \bar{\mathbf{J}}^1, \mathbf{f}^1, \bar{\mathbb{F}}^1) <^{\mathrm{at}}_{\mathfrak{u}} (\bar{M}^2, \bar{\mathbf{J}}^2, \mathbf{f}^2, \bar{\mathbb{F}}^2)$ with at standing for atomic, hold <u>when</u> both quadruples are from $K^{\mathrm{rt}}_{\mathfrak{u}}$ and there are a club E of ∂ and sequence $\bar{\mathbf{I}} = \langle \mathbf{I}_\alpha : \alpha < \partial \rangle$ witnessing it which means that we have

(*a*) $\delta \in E \Rightarrow \mathbf{f}^1(\delta) \leq \mathbf{f}^2(\delta)$ & $\mathrm{Min}(E \backslash (\delta + 1)) > \mathbf{f}^2(\delta)$

(*b*) for $\delta \in E$, if $i \leq \mathbf{f}^1(\delta)$ then $M^1_{\delta+i} \leq_{\mathfrak{K}_{<\partial}} M^2_{\delta+i}$ and if $i < \mathbf{f}^i(\delta)$ then $\mathbb{F}^1_{\delta+i} = \mathbb{F}^2_{\delta+i}$

(*c*) $\langle (M^0_\alpha, M^1_\alpha, \mathbf{I}_\alpha) : \alpha \in \cup\{[\delta, \delta + \mathbf{f}^1(\delta)] : \delta \in E\} \rangle$ is $\leq^1_{\mathfrak{u}}$-increasing continuous.

(*d*) if $\delta \in E$ and $i < \mathbf{f}^1(\delta)$ and A is the universe of $M^2_{\delta+i+1}$ <u>then</u> $(M^2_{\delta+i+1}, \mathbf{I}^2_{\delta+i+1}, \mathbf{J}^2_{\delta+i}) = \mathbb{F}^1_{\delta+i}(M^1_{\delta+i}, M^1_{\delta+i+1}, M^2_{\delta+i}, \mathbf{I}_{\delta+i}, \mathbf{J}^1_{\delta+i}, A)$.

4) We say that $(\bar{M}^\delta, \bar{\mathbf{J}}^\delta, \mathbf{f}^\delta, \bar{\mathbb{F}}^\delta)$ is a canonical upper bound of $\langle (\bar{M}^\alpha, \bar{\mathbf{J}}^\alpha, \mathbf{f}^\alpha, \bar{\mathbb{F}}) : \alpha < \delta \rangle$ as in Definition 1.15(4) adding: in clause (c), case 1 subclause (γ) to the conclusion $\mathbb{F}^\delta_{\zeta+i} = \mathbb{F}^{\alpha_\xi}_{\zeta+i}$ (and similarly in case 2).
4A) We say $(\langle \bar{M}^\alpha, \bar{\mathbf{J}}^\alpha, \mathbf{f}^\alpha, \bar{\mathbb{F}}^\alpha) : \alpha < \alpha(*) \rangle$ is a $<^{\mathrm{at}}_{\mathfrak{u}}$-tower <u>if</u>:

(*a*) $(\bar{M}^\alpha, \bar{\mathbf{J}}^\alpha, \mathbf{f}^\alpha, \bar{\mathbb{F}}^\alpha) <^{\mathrm{at}}_{\mathfrak{u}} (\bar{M}^{\alpha+1}, \mathbf{J}^{\alpha+1}, \bar{\mathbb{F}}^{\alpha+1})$ for when $\alpha + 1 < \alpha(*)$

(*b*) if $\delta < \alpha(*)$ is a limit ordinal, <u>then</u> $(\bar{M}^\delta, \bar{\mathbf{J}}^\delta, \mathbf{f}^\delta, \bar{\mathbb{F}}^\delta) \in K^{\mathrm{rt}}_{\mathfrak{u}}$ is a canonical upper bound of the sequence $\langle (\bar{M}^\alpha, \bar{\mathbf{J}}^\alpha, \mathbf{f}^\alpha, \mathbb{F}^\alpha) : \alpha < \delta \rangle$.

5) Let $(\bar{M}, \bar{\mathbf{J}}, \mathbf{f}, \bar{\mathbb{F}}) \leq^{\mathrm{rs}}_{\mathfrak{u}} (\bar{M}', \bar{\mathbf{J}}', \mathbf{f}', \bar{\mathbb{F}}')$ means that for some
$<^{\mathrm{at}}_{\mathfrak{u}}$-tower $\langle (\bar{M}^\alpha, \bar{\mathbf{J}}^\alpha, \mathbf{f}^\alpha, \bar{\mathbb{F}}^\alpha) : \alpha \leq \alpha(*) \rangle$ we have $(\bar{M}, \bar{\mathbf{J}}, \mathbf{f}, \bar{\mathbb{F}}) = (\bar{M}^0_{\alpha(*)}, \bar{\mathbf{J}}^0, \mathbf{f}^0, \bar{\mathbb{F}}^0)$ and $(\bar{M}', \bar{\mathbf{J}}', \mathbf{f}', \bar{\mathbb{F}}') = (\bar{M}^{\alpha(*)}, \bar{\mathbf{J}}^{\alpha(*)}, \mathbf{f}, \bar{\mathbb{F}}^{\alpha(*)})$.
6) We say that the sequence $\langle (\bar{M}^\alpha, \bar{\mathbf{J}}^\alpha, \mathbf{f}^\alpha, \bar{\mathbb{F}}^\alpha) : \alpha < \alpha(*)) \rangle$ is $\leq^{\mathrm{rs}}_{\mathfrak{u}}$-increasing continuous if it is $\leq^{\mathrm{rs}}_{\mathfrak{u}}$-increasing and for any limit $\delta < \alpha(*)$ the tuple $(\bar{M}^\delta, \bar{\mathbf{J}}^\delta, \mathbf{f}^\delta, \bar{\mathbb{F}}^\delta)$ is a canonical upper bound of the sequence of $\langle (\bar{M}^\alpha, \bar{\mathbf{J}}^\alpha, \mathbf{f}^\alpha, \bar{\mathbb{F}}^\alpha) : \alpha < \delta \rangle$.

10.4 Definition. For $\mathbb{F}$ a $\mathfrak{u}$-amalgamation choice function, see Definition 10.3(2):
1) $\mathbb{F}$ has uniqueness <u>when</u>:

$\circledast$ if $\mathbb{F}(M^\ell_0, M^\ell_1, M^\ell_2, \mathbf{J}^\ell_1, \mathbf{I}^\ell_1, A^\ell) = (M^\ell_3, \mathbf{J}^\ell_2, \mathbf{I}^\ell_2)$ for $\ell = 1, 2$ (all models standard) and f is a one to one function from A^1 onto A^2 preserving the order

[33]Dropping clause (d) causes little change

of the ordinals, f maps M^1_i to M^2_i for $i = 0,1,2$ (i.e. $f \restriction M^1_i$ is an isomorphism from M^1_i onto M^2_i) and $\mathbf{J}^1_1, \mathbf{I}^1_1$ onto $\mathbf{J}^2_1, \mathbf{I}^2_1$, respectively, <u>then</u> f is an isomorphism from M^1_3 onto M^2_3 mapping $\mathbf{J}^1_2, \mathbf{I}^1_2$ onto $\mathbf{J}^2_2, \mathbf{I}^2_2$ respectively.

2) $\mathbb{F}$ has strong uniqueness <u>when</u> $\mathbb{F}$ has uniqueness and

$\circledast$ if $\mathbb{F}(M^\ell_0, M^\ell_1, M^\ell_2, \mathbf{J}^\ell_1, \mathbf{I}^\ell_1, A^\ell) = (M^\ell_3, \mathbf{J}^\ell_2, \mathbf{I}^\ell_2)$ for $\ell = 1,2$ and f is a one to one mapping from $M^1_1 \cup M^1_2$ onto $M^2_1 \cup M^2_2$ such that: $f \restriction M^1_i$ is an isomorphism from M^1_i onto M^2_i for $i = 0,1,2$ and it maps $\mathbf{J}^1_1, \mathbf{I}^1_1$ onto $\mathbf{J}^2_1, \mathbf{I}^2_1$, respectively, <u>then</u> $|A^1 \backslash M^1_1 \backslash M^1_2| = |A^2 \backslash M^2_1 \backslash M^2_2|$, and there is an isomorphism g from M^1_3 onto M^2_3 extending f and mapping $\mathbf{J}^1_2, \mathbf{I}^1_2$ onto $\mathbf{J}^2_2, \mathbf{I}^2_2$ respectively; and moreover, $\mathrm{otp}(A^1 \backslash M^1_1, M^1_2) = \mathrm{otp}(A^2 \backslash M^2_1 \backslash M^2_2)$ and $f \restriction (A \backslash M^1_1 \backslash M^1_2)$ is order preserving.

10.5 Remark. 1) In Definition 10.3, in part (2) we can replace clause (e) by demanding A is an interval of the form $[\gamma_*, \gamma_* + \theta]$ where $i < \partial, \gamma_* = \cup\{\gamma + 1 : \gamma \in M_1$ or $\gamma \in M_2\}$. Then in part (3) we have M^1_∂ has universe δ for some $\delta < \partial^+$ and M^2_γ has universe $\delta + \partial$. Also the results in 10.7, 10.8 becomes somewhat more explicit.
2) We can fix $\mathbb{F}_*$, i.e. demand $\mathbb{F}_\alpha = \mathbb{F}_*$ in Definition 10.3.

10.6 Claim. *1) There is a $\mathfrak{u}$-amalgamation function with strong uniqueness.*
2) $K^{\mathrm{rt}}_{\mathfrak{u}}$ is non-empty, moreover for any stationary $S \subseteq \partial$ and triple $(\bar M, \bar{\mathbf{J}}, \mathbf{f}) \in K^{\mathrm{qt}}_{\mathfrak{u}}$, there is $\bar{\mathbb{F}}$ such that $(\bar M, \bar{\mathbf{J}}, \mathbf{f}, \bar{\mathbb{F}}) \in K^{\mathrm{rt}}_{\mathfrak{u}}$ with $S = \{\delta < \partial : \mathbf{f}(\delta) > 0\}$.
3) If $(\bar M, \bar{\mathbf{J}}^1, \mathbf{f}^1, \bar{\mathbb{F}}^1) \in K^{\mathrm{rt}}_{\mathfrak{u}}$ <u>then</u> for some $(\bar M^2, \bar{\mathbf{J}}^2, \mathbf{f}^2, \bar{\mathbb{F}}^2) \in K^{\mathrm{rt}}_{\mathfrak{u}}$ we have $(\bar M^1, \bar{\mathbf{J}}^1, \mathbf{f}^1, \bar{\mathbb{F}}^1)$ $<^{\mathrm{at}}_{\mathfrak{u}} (\bar M^2, \bar{\mathbf{J}}^2, \mathbf{f}^2, \bar{\mathbb{F}}^2)$; moreover, if δ is the universe of M^1_∂ then $\alpha < \partial \Rightarrow M^2_\alpha \backslash \delta \in \{[\delta, \delta + i) : i < \partial\}$.
4) Canonical upper bound as in 10.3(4) exists.

Proof. 1) Let $\mathbf{X}$ be the set of quintuples $\mathbf{x} = (M^{\mathbf{x}}_0, M^{\mathbf{x}}_1, M^{\mathbf{x}}_2, \mathbf{J}^{\mathbf{x}}_1, \mathbf{J}^{\mathbf{x}}_2)$ as in clause (a) of Definition 10.3(2). We define a two-place relation E on $\mathbf{X} : \mathbf{x} E \mathbf{y}$ iff $\mathbf{x}, \mathbf{y} \in \mathbf{X}$ and there is a one-to-one function f from $M^{\mathbf{y}}_1 \cup M^{\mathbf{y}}_2$ onto $M^{\mathbf{x}}_1 \cup M^{\mathbf{x}}_2$ such that $f \restriction M^{\mathbf{x}}_\ell$ is an isomorphism from $M^{\mathbf{x}}_\ell$ onto $M^{\mathbf{y}}_\ell$ for $\ell = 0,1,2$ and f maps $\mathbf{J}^{\mathbf{y}}_\ell$ onto $\mathbf{J}^{\mathbf{x}}_\ell$ for $\ell = 1,2$. Clearly E is an equivalence relation, and let $\mathbf{Y} \subseteq \mathbf{X}$ be a set of representatives and for every $\mathbf{x} \in \mathbf{X}$ let $\mathbf{y}(\mathbf{x})$ be the unique $\mathbf{y} \in \mathbf{Y}$ which is E-equivalent to it and let $f = f_{\mathbf{x}}$ be a one-to-one function from $M^{\mathbf{y}}_1 \cup M^{\mathbf{y}}_2$ onto $M^{\mathbf{x}}_1 \cup M^{\mathbf{x}}_2$ witnessing the equivalence.

For each $\mathbf{y} \in \mathbf{Y}$ by clause (F) of Definitin 1.2 there is a triple $(M^{\mathbf{y}}_3, \mathbf{J}^{+,\mathbf{y}}_1, \mathbf{J}^{+,\mathbf{y}}_2)$ such that

$(*)$ $(M^{\mathbf{x}}_0, M^{\mathbf{x}}_\ell, \mathbf{J}^{\mathbf{x}}_\ell) \le^\ell_{\mathfrak{u}} (M^{\mathbf{y}}_{3-\ell}, M^{\mathbf{y}}_3, \mathbf{J}^{+,\mathbf{y}}_\ell)$ for $\ell = 1,2$.

Without loss of generality $M_3^{\mathbf{y}}$ has universe $\subseteq \partial^+$, we can add has universe $M_1^{\mathbf{y}} \cup M_2^{\mathbf{y}} \cup [\gamma, \gamma+\theta)$ where $\gamma = \sup\{\alpha+1 : \alpha \in M_1^{\mathbf{y}} \cup M_2^{\mathbf{y}}\}$ where θ is the cardinality of $M_3^{\mathbf{y}} \backslash M_1^{\mathbf{y}} \backslash M_2^{\mathbf{y}}$.

Lastly, let us define $\mathbb{F}$ as follows:

- (a) $\zeta_{\mathbf{x}} = \mathrm{otp}(M_3^{\mathbf{y}(\mathbf{x})} \backslash M_1^{\mathbf{y}(\mathbf{x})} \backslash M_2^{\mathbf{y}(\mathbf{x})}) < \partial$ and
- (b) $\mathrm{Dom}(\mathbb{F}) = \{(\mathbf{x}, A) : \mathbf{x} \in \mathbf{X}, M_1^{\mathbf{x}} \cup M_2^{\mathbf{x}} \subseteq A \subseteq \partial$ and $\mathrm{otp}(A \backslash M_1^{\mathbf{x}} \backslash M_2^{\mathbf{x}}) = \zeta_{\mathbf{x}}\}$ where $(\mathbf{x}, A)$ means $(M_0^{\mathbf{x}}, M_1^{\mathbf{x}}, M_2^{\mathbf{x}}, \mathbf{J}_2^{\mathbf{x}}, \mathbf{J}_2^{\mathbf{x}}, A)$
- (c) for $(\mathbf{x}, A) \in \mathrm{Dom}(\mathbb{F})$ let $f_{\mathbf{x},A}$ be the unique one to one function from $M_3^{\mathbf{y}(\mathbf{x})}$ onto A which extends $f_{\mathbf{x}}$ and is order preserving mapping from $M_3^{\mathbf{y}(\mathbf{x})} \backslash M_1^{\mathbf{y}(\mathbf{x})} \backslash M_2^{\mathbf{y}(\mathbf{x})}$ onto $A \backslash M_1^{\mathbf{x}} \backslash M_2^{\mathbf{x}}$
- (d) for $(\mathbf{x}, A) \in \mathrm{Dom}(\mathbb{F})$ let $\mathbb{F}(\mathbf{x}, A)$ be the image under $f_{\mathbf{x},A}$ of $(M_3^{\mathbf{y}(\mathbf{x})}, \mathbf{J}_1^{+,\mathbf{y}(\mathbf{x})}, \mathbf{J}_2^{+,\mathbf{y}(\mathbf{x})})$.

Now check.
2) By part (1) and "$K_{\mathfrak{u}}^{\mathrm{qt}} \neq \emptyset$", see 1.19(1).
3) Put together the proof of 1.19(3) and part (2).
4) As in the proof of 1.19(4). $\square_{10.6}$.

10.7 Claim. *1) There is a function* $\mathbf{m}$ *satisfying:*

$\circledast_1$ *if* $(\bar{M}, \bar{\mathbf{J}}, \mathbf{f}, \bar{\mathbb{F}}) \leq_{\mathfrak{u}}^{\mathrm{rs}} (\bar{M}', \bar{\mathbf{J}}', \mathbf{f}', \bar{\mathbb{F}}')$, *recalling 10.3(5) then for some function* $h : \partial \rightarrow \mathscr{H}_{<\partial}(\partial^+)$, *but if* $2^\partial = \partial^+$ *then* $h : \partial \rightarrow \mathscr{H}_{<\partial}(\partial)$ *we have:*

$\odot$ *for a club of* $\delta < \partial$ *the object* $\mathbf{m}(h \restriction \delta, \bar{M}, \bar{\mathbf{J}}, \mathbf{f}, \bar{\mathbb{F}}, M'_\delta)$ *is a model* $N \in K_{\mathfrak{u}}$ *such that*

- (a) $M_{\delta+\mathbf{f}(\delta)} \leq_{\mathfrak{u}} N$
- (b) $M'_\delta \leq_{\mathfrak{u}} N$
- (c) *there is* $N' \leq_u M'_\partial$ *isomorphic to* N *over* $M_{\delta+\mathbf{f}(\delta)} + M'_\delta$ *in fact* $N' = M'_{\delta+\mathbf{f}(\delta)}$

$\circledast_2$ $\mathbf{m}$ *is preserved by partial, order preserving functions from* ∂^+ *to* ∂^+ *compatible with* id_{M_∂}

$\circledast_3$ *in fact in* $\odot$ *above,* $\mathbf{m}(h \restriction \delta, \bar{M}, \bar{\mathbf{J}}, \mathbf{f}, \bar{\mathbb{F}}, M'_\delta)$ *is actually* $\mathbf{m}(h \restriction \delta, \bar{M} \restriction [\delta, \delta + \mathbf{f}(\delta) + 1, \bar{\mathbf{J}} \restriction [\delta, \delta + \mathbf{f}(\delta)), \bar{\mathbb{F}} \restriction [\delta, \delta + \mathbf{f}(\delta)], M'_\delta)$.

Proof. By 10.8 we can define $\mathbf{m}$ explicitly. $\square_{10.7}$

10.8 Claim. $(\bar M', \bar{\mathbf J}', \mathbf f', \bar{\mathbb F}') \le^{\rm rs}_{\mathfrak u} (\bar M'', \bar{\mathbf J}'', \mathbf f'', \bar{\mathbb F}'')$ *<u>iff</u> both are from* $K^{\rm rt}_{\mathfrak u}$ *and we can find a* $E, \alpha(*) = \alpha_*, \bar u, \bar{\mathbf d}$ *such that:*

$\circledast$ (a) *E is a club of ∂*

(b) *α_* an ordinal $< \partial^+$*

(c) *$\bar u$ is a $\subseteq$-increasing continuous sequence $\langle u_i : i < \partial\rangle$*

(d) *$i < \partial \Rightarrow |u_i| < \partial$ and $\alpha_* + 1 = \cup\{\alpha_i : i < \partial\}$ and $\alpha \in u_0, 0 \in u_0$ and $(\forall \beta < \alpha)(\beta \in u_i \equiv \beta + 1 \in u_i)$*

(e) *$\bar{\mathbf d} = \langle \mathbf d_\delta : \delta \in E\rangle$*

(f) *$\mathbf d_\delta$ is a $\mathfrak u$-free $(\mathbf f(\delta), \mathrm{otp}(u_\delta))$-rectangle, see Definition 1.4*

(g) *there is a $\le^{\rm at}_{\mathfrak u}$-tower $\langle(\bar M^\alpha, \bar{\mathbf J}^\alpha, \mathbf f^\alpha, \bar{\mathbb F}^\alpha) : \alpha \le \alpha_*\rangle$ as in 10.3(6) witnessing the assumption and $\langle \bar{\mathbf I}^\alpha : \alpha < \alpha_*\rangle$ with $\mathbf I^\alpha$ witnessing $(\bar M^\alpha, \bar{\mathbf J}^\alpha, \mathbf f^\alpha, \bar{\mathbb F}^\alpha) <^{\rm at}_{\mathfrak u_*} (\bar M^{\alpha+1}, \bar{\mathbf J}^{\alpha+1}, \mathbf f^{\alpha+1}, \bar{\mathbb F}^{\alpha+1})$ (so $(\bar M^0, \bar{\mathbf J}^0, \mathbf f^0, \bar{\mathbb F}^0) = (\bar M', \bar{\mathbf J}', \mathbf f', \bar{\mathbb F}'), (\bar M^{\alpha(*)}, \bar{\mathbf J}^{\alpha(*)}, \mathbf f^{\alpha(*)}, \bar{\mathbb F}^{\alpha(*)}) = (\bar M'', \bar{\mathbf J}'', \mathbf f'', \bar{\mathbb F}''))$ such that*

$\boxdot$ *if $\beta \in [0, \alpha_* + 1), \delta \in E$ and $\beta \in u_\delta$ and $j = \mathrm{otp}(\beta \cap u_\delta)$ <u>then</u> for every $i \le \mathbf f(\delta)$*

(α) $M^{\mathbf d_\delta}_{i,j} = M^\beta_{\delta+i}$

(β) $\mathbf J^{\mathbf d_\delta}_{i,0} = \mathbf J'_{\delta+i}$ *when $i < \mathbf f(\delta)$*

(γ) $\mathbf I^{\mathbf d_\delta}_{0,j} = \mathbf I^\beta_\delta$ *when $\beta < \alpha_*$*

(δ) *if $i < \mathbf f(\delta)$ and $\beta < \alpha_*$ (so $\mathrm{otp}(\beta \cap u_\delta) < \mathrm{otp}(u_\delta)$) then $(M^{\mathbf d_\delta}_{i+1,j+1}, \mathbf I^{\mathbf d_\delta}_{i,j+1}, \mathbf J^{\mathbf d_\delta}_{i+1,j}) = \mathbb F_{\delta+i}(M^{\mathbf d_\delta}_{i,j}, M^{\mathbf d_\delta}_{i+1,j}, M^{\mathbf d_\delta}_{i,j+1}, \mathbf I^{\mathbf d_\delta}_{i,j}, \mathbf J^{\mathbf d_\delta}_{i,j}, |M^{\mathbf d_\delta}_{i+1,j+1}|)$.*

Proof. Straight. $\square_{10.8}$

10.9 Remark. If we define the version of 10.3(3) with $|M^2| = |M^1| + \partial$ <u>then</u> $\mathbf m(-)$ is O.K. not only up to isomorphism but really given the value.

10.10 Claim. *Theorem 2.3 holds.*

That is, $\dot I_\tau(\partial^+, K^{\mathfrak u,\mathfrak h}_{\partial^+}) \ge \mu_{\rm unif}(\partial^+, 2^\partial)$ <u>when</u>:

$\circledast$ (a) $2^\theta = 2^{<\partial} < 2^\partial$

(b) $2^\partial < 2^{\partial^+}$

(c) *the ideal* WDmId(∂) *is not* ∂^+*-saturated*

(d) $\mathfrak{u}$ *has the weak* τ*-coding, see Definition 2.2(5) (or just above some triple* $(\bar M^*, \bar{\mathbf{J}}^*, \mathbf{f}^*) \in K^{\rm qt}_{\mathfrak{u}}$ *with* WDmId$(\partial) \restriction (\mathbf{f}^*)^{-1}\{0\}$ *not* ∂^+*-saturated)*

(e) $\mathfrak{h}$ *is* $\mathfrak{u} - \{0,2\}$*-appropriate.*

10.11 Remark. 1) Similarly 2.7 holds.
2) We can below (in $\boxtimes$) immitate the proof of 3.3.

Proof. Clearly when $(\bar M^*, \bar{\mathbf{J}}^*, \mathbf{f}^*)$ as in clause (d) is not given, by 10.6(2) we can choose it, even with $\mathbf{f}^*$ constantly zero, so without loss of generality such a triple is given. By 1.25(4) and clauses (d) + (e), without loss of generality :

$\otimes$ $\mathfrak{h} = (\mathfrak{h}_0, \mathfrak{h}_2)$ witness that $\{0,2\}$-almost every triple $(\bar M, \bar{\mathbf{J}}, \mathbf{f}) \in K^{\rm qt}_{\mathfrak{u}}$ above $(\bar M^*, \bar{\mathbf{J}}^*, \mathbf{f}^*)$ has the weak coding property.

Let $\bar S$ be such that

$\odot$ (a) $\bar S = \langle S^*_\zeta : \zeta < \partial^+ \rangle$

(b) $S^*_\zeta \subseteq \partial$

(c) S^*_ζ is increasing modulo $[\partial]^{<\partial}$

(d) S^*_0 and $S^*_{\zeta+1} \backslash S^*_\zeta \notin$ WDmId(∂)

(e) $\mathbf{f}^* \restriction (\partial \backslash S^*_0)$ is constantly zero.

Such sequence exists by clause (c) of the hypothesis. It suffices to deal with the case $\mathfrak{h}_2$ is a $\mathfrak{u} - 2 - S^*_0$-appropriate function, see Definition 1.24(2A).

We choose $\langle (\bar M^\eta, \bar{\mathbf{J}}^\eta, \mathbf{f}^\eta, \bar{\mathbb{F}}^\eta) : \eta \in {}^\gamma(2^\partial) \rangle$ by induction on $\gamma < \partial^+$ such that:

$\boxplus$ (a) $(\bar M^\eta, \bar{\mathbf{J}}^\eta, \mathbf{f}^\eta, \bar{\mathbb{F}}^\eta) \in K^{\rm rt}_{\mathfrak{u}}$ and if $\gamma = 0$ then $(\bar M^\eta, \bar{\mathbf{J}}^\eta, \mathbf{f}^\eta) = (\bar M^*, \bar{\mathbf{J}}^*, \mathbf{f}^*)$

(b) if $i \notin S^*_{\ell g(\eta)}$ then $\mathbf{f}_\eta(i) = 0$

(c) $\langle (\bar M^{\eta \restriction \beta}, \bar{\mathbf{J}}^{\eta \restriction \beta}, \mathbf{f}^{\eta \restriction \beta}, \bar{\mathbb{F}}^{\eta \restriction \beta} : \beta \le \gamma \rangle$ is $\le^{\rm rs}_{\mathfrak{u}}$-increasing continuous,

(d) if $\eta \in {}^{\gamma+1}(2^\partial)$ and γ is a non-limit ordinal and $\alpha < 2^\partial$ <u>then</u> the pair $((\bar M^\eta, \bar{\mathbf{J}}^\eta, \mathbf{f}^\eta), (\bar M^{\eta\hat{}<\alpha>}, \bar{\mathbf{J}}^{\eta\hat{}<\gamma>}, \mathbf{f}^{\eta\hat{}<\alpha>}))$ strictly $1 - S^*_0$-obey $\mathfrak{h}_2$ and 0-obey $\mathfrak{h}_0$ (see Definition 1.22(1), 1.24(2)
so without loss of generality for all $\eta\hat{}\langle\alpha\rangle$ we choose the same value

(e) $\eta \in {}^\gamma(2^\partial)$ and $\alpha_1 \ne \alpha_2 < 2^\partial, \gamma$ a limit ordinal (even ∂ divides γ) and $(\bar M^{\eta\hat{}<\alpha_2>}, \bar{\mathbf{J}}^{\eta\hat{}<\alpha_2>}, \bar{\mathbb{F}}^{\eta\hat{}<\alpha_2>}) \le^{\rm rt}_{\mathfrak{u}} (M', \bar{\mathbf{J}}', \mathbf{f}', \bar{\mathbb{F}}')$ <u>then</u> $M^{\eta\hat{}<\alpha_1>}_\partial$ cannot be $\le_{\mathfrak{K}}$-embedded into M'_∂ over M_η.

Why this is enough? By 9.1, noting that

(∗) if $\gamma(*) < \partial^+$ and $\eta\char`\^\langle\alpha_i\rangle \trianglelefteq \nu_i \in {}^{\gamma(*)}(2^\partial)$ for $i = 0, 1$ and $\alpha_0 < \alpha_1 < 2^\partial$ and $f$ is an isomorphism from $M^{\nu_0}_\partial$ onto $M^{\nu_1}_\partial$ over $M^\eta_\partial$ <u>then</u> $\eta, f \restriction M^{\eta\char`\^<\alpha_0>}_\partial, (\bar M_{\nu_1}, \bar{\mathbf{J}}_{\nu_1}, \mathbf{f}_{\nu_2}, \bar{\mathbb{F}}_{\nu_1})$ form a counterexample to clause (e) of ⊞.

For $\gamma = 0$ clause (a) of ⊞, i.e. choose $(\bar M^*, \bar{\mathbf{J}}^*, \mathbf{f}^*)$ recallng our use of 10.6(2).

For γ limit use 10.6(4).

So assume $\eta \in {}^{\gamma(*)}(2^\partial)$ and $(\bar M^\eta, \bar{\mathbf{J}}^\eta, \mathbf{f}^\eta, \bar{\mathbb{F}}^\eta)$ has been defined and we should deal with $\Xi_\eta := \{\eta\char`\^\langle\alpha\rangle : \alpha < 2^\partial\}$.

We choose $(\alpha(0), N_0, \mathbf{I}_0)$ such that

⊕ (a) $\alpha(0) < \partial$

(b) $(M^\eta_{\alpha(0)}, N_0, \mathbf{I}_0) \in$ FR$_1$

(c) $N_0 \cap M^\eta_\partial = M^\eta_{\alpha(0)}$

(d) if $\gamma(*)$ is a non-limit ordinal then $(\alpha(0), N, \mathbf{I})$ is as dictated by $\mathfrak{h}$, i.e. $\mathfrak{h}_0$, see Definition 1.24(1)(c)

(e) if $\gamma(*)$ is a limit ordinal and $(\bar M^\eta, \bar{\mathbf{J}}^\eta, \mathbf{f}^\eta)$ has the weak coding$_1$ property (see Definition 2.2(3)) <u>then</u> for a club of $\alpha(1) \in (\alpha(0), \gamma)$ we have:

(∗) if $(M_{\alpha(0)}, N_0, \mathbf{I}_0) \le^1_{\mathfrak{u}} (M_{\alpha(1)}, N^*_1, \mathbf{I}^*_1)$ and $M^\eta_\partial \cap N^*_1 = M^\eta_{\alpha(1)}$ <u>then</u> there are $\alpha(2) \in (\alpha(1), \gamma)$ and $N^\ell_2, \mathbf{I}^\ell_2$ for $\ell = 1, 2$ such that: $(M^\eta_{\alpha(1)}, N_1, \mathbf{I}_1) \le^1_{\mathfrak{u}} (M^\eta_{\alpha(2)}, N^\ell_2, \mathbf{I}^\ell_2)$ for $\ell = 1, 2$ and N^1_2, N^2_2 are τ-incompatible amalgamations of $M^\eta_{\alpha(2)}, N_1$ over $M^\eta_{\alpha(1)}$.

Without loss of generality

(f) $|N_0| \backslash M^\eta_{\alpha(0)}$ is an initial segment of $\partial^+ \backslash |M^\eta_\partial|$.

We shall use 9.3. Toward this we choose u_i and if $i \in u_i$ also $E^\eta_\rho, M^\eta_\rho, \mathbf{I}^\eta_\rho, \mathbf{J}^\eta_\rho$ (but $\mathbf{J}^\eta_\rho$ is chosen in the $(i+1)$-th step) for $\rho \in {}^i 2$ induction on $i \in [\alpha(0), \partial)$ such that:

⊠$_1$ (a) $u_i \subseteq [\alpha(0), i]$ is closed

(b) if $j < i$ then $u_i \cap (j+1) = u_j$

(c) E^η_ρ is a closed subset of $i \cap u_i$

(d) $j < \ell g(\rho) = E^\eta_{\rho \restriction j} = E^\eta_\rho \cap j$

(e) if $j \in E^\eta_\rho$ then $\mathbf{f}_\eta(j) < \min\{(E^\eta_\rho \backslash j)$ or $E^\eta_\rho \subseteq j\}$ and $(j, j+f^\eta(j)] \subseteq u_i$

(f) $M^\eta_\rho \in K^{\mathfrak{u}}_{<\partial}$ and $M^\eta_\rho \cap M^\eta = M^\eta_{\ell g(\rho)}$

(g) $\langle M^\eta_{\rho \restriction j} : j \in u_i \rangle$ is $\leq_{\mathfrak{K}_{<\partial}}$-increasing continuous

(h) $\langle (M^\eta_j, M^\eta_{\rho\restriction j}, \mathbf{I}^\eta_\rho) : j \in u_i \rangle$ is $\leq^1_{\mathfrak{u}}$-increasing continuous

(i) (α) $(M^\eta_j, M^\eta_{j+1}, \mathbf{J}^\eta_j) \leq^2_{\mathfrak{u}} (M^\eta_{\rho\restriction j}, M^\eta_{\rho\restriction(j+1)}, \mathbf{J}^\eta_{\rho\restriction j})$ when $j \in i \cap u_i$

(β) if $j \in \cup\{[\zeta, \zeta + \mathbf{f}^\eta(\zeta)) : \zeta \in E^\eta_\rho\}$ then moreover we get $(M^\eta_{\rho\restriction(j+1)}, \mathbf{I}^\eta_{\rho\restriction j}, \mathbf{J}^\eta_{\rho\restriction j})$ by applying $\mathbb{F}^\eta_j$

(j) if $i = \ell g(\rho) = j+1$, j is limit $\in S^*_{\gamma(*)+1} \backslash S^*_{\gamma(*)}$ and $\cup\{E_{\rho\restriction j} : j < \ell g(\rho)\}$ is unbounded in $\ell g(\rho)$ and $\mathbf{f}_\eta(\ell g(\rho)) = 0$ and if we can <u>then</u> $u_i = u_j \cup \{i\}$ and $(M^\eta_{\rho\char`^<\ell>}, \mathbf{I}^\eta_{\rho\char`^<\ell>}, \mathbf{J}^\eta_{\rho\char`^<\ell>})$ for $\ell = 0,1$ are gotten as in $\oplus$ above so in particular $M^\eta_{\rho\char`^<0>}, M^\eta_{\rho\char`^<\ell>}$ are τ-incompatible amalgamations of M^η_i, M^η_ρ over M^η_j

(k) if $i = j+1, \delta = \max(E^\eta_{\rho\restriction j}) \leq i, \delta \in S^*_0, j = \delta + \mathbf{f}^\eta(\delta)$, <u>then</u> we act as dictated by $\mathfrak{h}$, i.e. $\mathfrak{h}_2$; moreover this holds for all the interval $[\delta + \mathbf{f}^\eta(\delta), \delta + \mathbf{f}^\eta(\delta) + i']$ for an appropriate $i' < \partial$ by the "dictation" of $\mathfrak{h}_2$ (see Definition 1.22).

In clause (j) this is possible for enough times, if $(\bar{M}^\eta, \bar{\mathbf{J}}^\eta, \mathbf{f}^\eta)$ has the weak coding$_1$ property, i.e. for $\rho \in {}^\partial 2$, for a club of $i \in \mathbf{f}^{-1}_\eta\{0\}$ by the choice of $\mathfrak{h}$. Trace the Definitions.
Also $(\bar{M}^\eta, \bar{\mathbf{J}}^\eta, \mathbf{f}^\eta)$ has the weak coding property by the choice of $\mathfrak{h}$ and the induction hypothesis.

Clearly we can carry the induction on $i < \lambda$ and by 9.3 carrying the induction to $\gamma(*)+1$, so we have finished carrying the induction. So by 9.1 we are done. $\square_{10.10}$

10.12 Claim. *Theorem 2.11 holds.*
That is, $\dot{I}_\tau(\partial^+, K^{\mathfrak{u}}_{\partial^+}) \geq \mu_{\text{unif}}(\partial^+, 2^\partial)$, *<u>when</u>:*

$\circledast$ (a) $2^\theta = 2^{<\partial} < 2^\partial$

(b) $2^\partial < 2^{\partial^+}$

(c) $\mathfrak{u}$ *has the vertical* τ*-coding*$_1$ *property above some triple from* $K^{\text{qt}}_{\mathfrak{u}}$.

Proof. Like the proof of 10.10 but:

<u>Change (A)</u>: We omit $\bar{S}$, i.e. $\odot$, the choice of $\langle S^*_\varepsilon : \varepsilon < \partial^+ \rangle$, can use $S^*_0 = \partial$, or more transparently, use $S^*_\zeta = S^*_0$ stationary, $\partial \backslash S^*_\zeta \in (\text{WDmId}(\partial))^+$, on S^*_0 act as before on $\partial \backslash S^*_0$ and act as on $S^*_{\zeta+1} \backslash S^*_\zeta$ before.

Change (B): We change clause $\boxtimes(j)$ to fit the present coding so any limit ordinal j. $\square_{10.12}$

10.13 Claim. *Theorem 2.15 holds.*
That is $\dot{I}_\tau(\partial^+, K^{\mathfrak{u}}_{\partial^+}) \ge 2^{\partial^+}$ *when:*

(a) $2^\theta = 2^{<\partial} < 2^\partial$
(b) $2^\partial < 2^{\partial^+}$
(c) $\mathfrak{u}$ *has horizontal τ-coding property, say just above* $(\bar{M}^*, \bar{\mathbf{J}}^*, \mathbf{f}^*)$
(d) *the ideal* WDmId(∂) *is not ∂^+-saturated.*

Proof. Similar to the proof of 10.10 but:

Change (A): We omit $\odot$, i.e. $\bar{S}^*$ and use $S^*_0 = \partial$

Change (B): In $\boxplus$ we use only $\eta \in {}^{\partial^+}2$ and clause (e) is changed to:

(e)″ if $(\bar{M}^{\eta\char`^<1>}, \bar{\mathbf{J}}^{\eta\char`^<1>}, \mathbf{f}^{\eta\char`^<1>}, \bar{\mathbb{F}}^{\eta\char`^<1>}) \le^{\mathrm{rt}}_{\mathfrak{u}} (\bar{M}', \bar{\mathbf{J}}', \mathbf{f}', \bar{\mathbb{F}}')$ then $M^{\eta\char`^<0>}_\partial$ cannot be $\le_{\mathfrak{K}}$-embedded into M'_∂ over $M^{<>}_\partial$.

Change (C): We change $\boxtimes$ clause (j) to deal with the present coding.

Change (D): We use 9.5 rather than 9.1. $\square_{10.13}$

* * *

10.14 Discussion: 1) Instead constructing $\le^{\mathrm{at}}_{\mathfrak{u}}$-successors $(\bar{M}^{\eta\char`^\langle\alpha\rangle}, \mathbf{J}^{\eta\char`^\langle\alpha\rangle}, \mathbf{f}^{\eta\char`^\langle\alpha\rangle})$ of $(\bar{M}^\eta, \bar{\mathbf{J}}^\eta, \mathbf{f}^\eta)$, we may like to build, for each $\alpha < 2^\partial$ an increasing sequence of length ζ, first with $\zeta < \partial$ then even $\zeta < \partial^+$ but a sequence of approximations of height ∂.

We would like to have in quite many limit $\delta < \partial$ a "real choice" as the various coding properties says. How does this help? If arriving to $\eta \in {}^\delta(2^\partial), \delta < \partial^+, \eta\char`^\langle\alpha\rangle$, the model $M^{\eta\char`^<\alpha>}_\delta$ is brimmed over M_δ; this is certainly beneficial and having a tower arriving to δ help toward this. But it has a price - we have to preserve it. In case we have existence for $K^{3,\mathrm{up}}_{\mathfrak{s}}$ this occurs, see the proof of Theorem 8.14 (but was proved in an ad-hoc way).

2) So we have a function ι such that; so during the construction, for $\eta \in {}^{\partial^+}(2^\partial)$ letting $S_\eta := \{\xi < \ell g(\eta) : \langle(\bar{M}^{(\eta\restriction\xi)\char`^<\alpha>}, \bar{\mathbf{J}}^{(\eta\restriction\xi)\char`^<\alpha>}, \mathbf{f}^{(\eta\restriction\xi)\char`^<\alpha>}) : \alpha < 2^\partial\rangle$ is not constant$\}$ we have:

(a) if $\xi = \ell g(\eta) = \sup(S_\eta)+1$ and $\iota = \iota(\bar{M}^\eta \restriction (\xi+1), \bar{\mathbf{J}}^{\eta\restriction(\xi+1)}, f^{\eta\restriction(\xi+1)}), \bar{\mathbb{F}}^{\eta\restriction(\xi+1)})$ and $\eta \triangleleft \nu_\ell \in {}^{\ell g(\eta)+\iota}(2^\partial)$ for $\ell = 1,2$ then $(\bar{M}^{\nu_1}, \bar{\mathbf{J}}^{\nu_1}, \mathbf{f}^{\nu_1}, \bar{\mathbb{F}}^{\nu_1}) = (\bar{M}^{\nu_2}, \bar{\mathbf{J}}^{\nu_2}, \mathbf{f}^{\nu_2}, \mathbb{F}^{\nu_2})$.

To formalize this we can use (see a concrete example in the proof in 8.17).

10.15 Definition. 1) We say $\mathbf{i}$ is a ∂-parameter <u>when</u>:

- (a) $\mathbf{i} = (\iota, \bar{u})$
- (b) ι is an ordinal ≥ 1 but $< \partial^+$
- (c) $\bar{u} = \langle u_\varepsilon : \varepsilon < \partial \rangle$ is a $\subseteq$-increasing sequence of subsets of ι of cardinality $< \partial$ with union ι
- (d) if δ is a limit ordinal $< \partial$ then u_δ is the closure of $\cup\{u_\varepsilon : \varepsilon < \delta\}$.

2) We say $\mathbf{d}$ is a $\mathfrak{u}$-free $([\varepsilon_1, \varepsilon_2], \mathbf{i})$-rectangle <u>when</u>:

- (a) $\varepsilon_1 \leq \varepsilon_2 < \partial$
- (b) $\mathbf{d}$ is a $\mathfrak{u}$-free $([\varepsilon_1, \varepsilon_2], \iota)$-rectangle

(so ι may be $\geq \partial$, but then not serious; in fact, it is an $\mathfrak{u}$-free $([\varepsilon_1, \varepsilon_2], u_\delta)$-rectangle but we complete it in the obvious way.)
3) The short case is when $\mathbf{i}$ is short, i.e. $\iota = 1, u_\alpha = 1$.
The long case is $\iota = \partial, u_\varepsilon = \varepsilon + 1$.

<u>10.16 Discusssion</u>: 1) Above we have concentrated on what we may call the "short" case, the "long" case as described in 10.14, 10.15 allows more constructions by "consuming" more levels.
2) Above we can restrict ourselves to the case $\partial = \lambda^+$ so in 10.3 we then demand on $(A \backslash M_1 \backslash M_2)$ is just "equal to λ" and the possible variants of 10.3(2),(a) + (b)(B) are irrelevant.

§11 Remarks on pcf

This section will provide us two pcf claims we use. One is 11.1, a set-theoretic division into cases when $2^\lambda < 2^{\lambda^+}$ (it is from pcf theory; note that the definition of WDmId(λ) is recalled in 0.3(4)(b) = 0.3(4)(b) and of $\mu_{\rm wd}(\lambda)$ is recalled in 0.3(8) = 0.3(8)), we can replace λ^+ by regular λ such that $2^\theta = 2^{<\lambda} < 2^\lambda$ for some θ). The second deals with the existence of large independent subfamilies of sets, 11.4. This is a revised version of a part of [Sh 603]. See on history related to 11.1 in [Sh:g] particularly in [Sh:g, II,5.11] and [Sh 430].

Remark. Recall that

$$\begin{aligned}\text{cov}(\chi, \mu, \theta, \sigma) = \chi + \ \text{Min}\big\{|\mathscr{P}| : &\mathscr{P} \subseteq [\chi]^{<\mu} \text{ and every member of} \\ &[\chi]^{<\theta} \text{ is included in the union of } < \sigma \text{ members of } \mathscr{P}\big\}.\end{aligned}$$

11.1 Claim. *Assume* $2^\lambda < 2^{\lambda^+}$.

Then one of the following cases occurs: (clauses $(\alpha) - (\lambda)$ *appear later)*

- $(A)_\lambda$ $\chi^* = 2^{\lambda^+}$ *and for some* μ *clauses* $(\alpha) - (\varepsilon)$ *hold*
- $(B)_\lambda$ *for some* $\chi^* > 2^\lambda$ *and* μ *clauses* $(\alpha) - (\kappa)$ *hold (note:* μ *appear only in* $(\alpha) - (\varepsilon)$*)*
- $(C)_\lambda$ $\chi^* = 2^\lambda$ *and clauses* $(\eta) - (\mu)$ *hold*

  where

  - (α) $\lambda^+ < \mu \leq 2^\lambda$ *and* $\mathrm{cf}(\mu) = \lambda^+$
  - (β) $\mathrm{pp}(\mu) = \chi^*$, *moreover* $\mathrm{pp}(\mu) =^+ \chi^*$
  - (γ) $(\forall \mu')(\mathrm{cf}(\mu') \leq \lambda^+ < \mu' < \mu \Rightarrow \mathrm{pp}(\mu') < \mu)$ *hence* $\mathrm{cf}(\mu') \leq \lambda^+ < \mu' < \mu \Rightarrow \mathrm{pp}_{\lambda^+}(\mu') < \mu$
  - (δ) *for every regular cardinal* χ *in the interval* $(\mu, \chi^*]$ *there is an increasing sequence* $\langle \lambda_i : i < \lambda^+ \rangle$ *of regular cardinals* $> \lambda^+$ *with limit* μ *such that* $\chi = \mathrm{tcf}\left(\prod\limits_{i<\lambda^+} \lambda_i / J^{\mathrm{bd}}_{\lambda^+}\right)$, *and* $i < \lambda^+ \Rightarrow \max \mathrm{pcf}\{\lambda_j : j < i\} < \lambda_i < \mu$
  - (ε) *for some regular* $\kappa \leq \lambda$, *for any* $\mu' < \mu$ *there is a tree* $\mathscr{T}$ *with* $\leq \lambda$ *nodes,* κ *levels and* $|\lim_\kappa(\mathscr{T})| \geq \mu'$ *(in fact e.g.* $\kappa = \mathrm{Min}\{\theta : 2^\theta \geq \mu\}$ *is appropriate; without loss of generality* $\mathscr{T} \subseteq {}^{\kappa >}\lambda$*)*
  - (ζ) *there is no normal* λ^{++}*-saturated ideal on* λ^+
  - (η) *there is* $\langle \mathscr{T}_\zeta : \zeta < \chi^* \rangle$ *such that:* $\mathscr{T}_\zeta \subseteq {}^{\lambda^+ >}2$, *a subtree of cardinality* λ^+ *and* ${}^{\lambda^+}2 = \{\lim_{\lambda^+}(\mathscr{T}_\zeta) : \zeta < \chi^*\}$
  - (θ) $\chi^* < 2^{\lambda^+}$ *moreover* $\chi^* < \mu_{\mathrm{unif}}(\lambda^+, 2^\lambda)$, *but* $< \mu_{\mathrm{unif}}(\lambda^+, 2^\lambda)$ *is not used here,*
  - (ι) *for some* $\zeta < \chi^*$ *we have* $\lim_{\lambda^+}(\mathscr{T}_\zeta) \notin \mathrm{UnfmTId}_{(\chi^*)^+}(\lambda^+)$, *not used here*
  - (κ) $\mathrm{cov}(\chi^*, \lambda^{++}, \lambda^{++}, \aleph_1) = \chi^*$ *or* $\chi^* = \lambda^+$, *equivalently* $\chi^* = \sup[\{\mathrm{pp}(\chi) : \chi \leq 2^\lambda, \aleph_1 \leq \mathrm{cf}(\chi) \leq \lambda^+ < \chi\} \cup \{\lambda^+\}]$ *by [Sh:g, Ch.II,5.4]; note that clause* (κ) *trivially follows from* $\chi^* = 2^{\lambda^+}$
  - (λ) *for no* $\mu \in (\lambda^+, 2^\lambda]$ *do we have* $\mathrm{cf}(\mu) \leq \lambda^+$, $\mathrm{pp}(\mu) > 2^\lambda$*; equivalently* $2^\lambda > \lambda^+ \Rightarrow \mathrm{cf}([2^\lambda]^{\lambda^+}, \subseteq) = 2^\lambda$
  - (μ) *if there is a normal* λ^{++}*-saturated ideal on* λ^+, *moreover the ideal* $\mathrm{WDmId}(\lambda^+)$ *is, then* $2^{\lambda^+} = \lambda^{++}$ *(so as* $2^\lambda < 2^{\lambda^+}$ *clearly* $2^\lambda = \lambda^+$*).*

Proof. This is related to [Sh:g, II,5.11]; we assume basic knowledge of pcf (or a readiness to believe quotations). Note that by their definitions

> $\circledast_1$ if $2^\lambda > \lambda^+$ then for any $\theta \in [\aleph_0, \lambda^+]$ we have $\mathrm{cf}([2^\lambda]^{\le \lambda^+}, \subseteq) = 2^\lambda \Leftrightarrow$ $\mathrm{cov}(2^\lambda, \lambda^{++}, \lambda^{++}, 2) = 2^\lambda \Leftrightarrow \mathrm{cov}(2^\lambda, \lambda^{++}, \lambda^{++}, \theta) = 2^\lambda$.

[Why? Because $(2^\lambda)^{<\lambda^+} = 2^\lambda$ and $\mathrm{cf}([2^\lambda]^{\le\lambda^+}, \subseteq) = \mathrm{cov}(2^\lambda, \lambda^{++}, \lambda^{++}, 2)$ $\ge \mathrm{cov}(2^\lambda, \lambda^{++}, \lambda^{++}, \aleph_0) \ge \mathrm{cov}(2^\lambda, \lambda^{++}, \lambda^{++}, \theta) \ge 2^\lambda$ for $\theta \in [\aleph_0, \lambda]$.]

Note also that

> $\circledast_2$ $\lambda^+ \notin \mathrm{WDmId}(\lambda^+)$ and ${}^{\lambda^+}2 \notin \mathrm{WDmTId}(\lambda^+)$.

[Why? Theorem 0.5(2) with θ, ∂ there standing for λ, λ^+ here.]

Possibility 1: $2^\lambda > \lambda^+$ and $\mathrm{cov}(2^\lambda, \lambda^{++}, \lambda^{++}, \aleph_1) = 2^\lambda$ or $2^\lambda = \lambda^+$; and let $\chi^* := 2^\lambda$.

We shall show that case (C) holds (for the cardinal λ), the first assertion "$\chi^* = 2^\lambda$" holds by our choice.

Now clause (κ) is obvious. As for clause (η), we have $\chi^* = 2^\lambda < 2^{\lambda^+}$. Now if $2^\lambda = \lambda^+$ we let $\mathscr{T}_\zeta = {}^{\lambda^+ >}2$, for $\zeta < \chi^*$ so clause (η) holds, otherwise as ${}^{\lambda^+>}2$ has cardinality 2^λ, by the definitions of $\mathrm{cov}(2^\lambda, \lambda^{++}, \lambda^{++}, \aleph_1)$ and the possibility assumption (and obvious equivalence) there is $\mathscr{P} \subseteq [{}^{\lambda^+>}2]^{\lambda^+}$ of cardinality χ^* such that any $A \in [{}^{\lambda^+>}2]^{\lambda^+}$ is included in the union of $\le \aleph_0$ of them. So $\eta \in {}^{\lambda^+}2 \Rightarrow (\exists A \in \mathscr{P})(\exists^{\lambda^+} \alpha < \lambda^+)(\eta \restriction \alpha \in A)$ so let $\langle A_\zeta : \zeta < \chi^* \rangle$ list $\mathscr{P}$ and let $\mathscr{T}_\zeta = \{\eta \restriction \alpha : \eta \in A_\zeta \text{ and } \alpha \le \ell g(\eta)\}$, now check that they are as required in clause (η).

As on the one hand by [Sh:f, AP,1.16 + 1.19] or see 9.4 we have $\big(\mu_{\mathrm{unif}}(\lambda^+, 2^\lambda)\big)^{\aleph_0} = 2^{\lambda^+} > 2^\lambda = \chi^*$ and on the other hand $(\chi^*)^{\aleph_0} = (2^\lambda)^{\aleph_0} = 2^\lambda = \chi^*$ necessarily $\chi^* < \mu_{\mathrm{unif}}(\lambda^+, 2^\lambda)$ so clause (θ) follows; next clause (ι) follows from clause (η) by the definition of $\mathrm{UnfTId}_{(\chi^*)^+}(\lambda^+)$. In fact in our possibility for some ζ, $\lim_{\lambda^+}(\mathscr{T}_\zeta) \notin \mathrm{WDmTId}(\lambda^+)$ because $\mathrm{WDmTId}(\lambda^+)$ is $(2^\lambda)^+$-complete by 0.5(2),(4) recalling $\circledast_2$ and having chosen $\chi^* = 2^\lambda$.

Now if $2^{\lambda^+} > \lambda^{++}$, (so $2^{\lambda^+} \ge \lambda^{+3}$), then for some $\zeta < \chi^*, \mathscr{T}_\zeta$ is (a tree with $\le \lambda^+$ nodes, λ^+ levels and) at least λ^{+3} λ^+-branches which is well known (see e.g. [J]) to imply "no normal ideal on λ^+ is λ^{++}-saturated"; so we got clause (μ). Also if $2^{\lambda^+} \le \lambda^{++}$ then $2^\lambda = \lambda^+, 2^{\lambda^+} = \lambda^{++}$.

As for clause (λ), by the definition of χ^* and the assumption $\chi^* = 2^\lambda$ we have the first two phrases. The "equivalently" holds as $(2^\lambda)^{\aleph_0} = 2^\lambda$.

Possibility 2: $\chi^* := \mathrm{cov}(2^\lambda, \lambda^{++}, \lambda^{++}, \aleph_1) > 2^\lambda > \lambda^+$.

So $(\mathrm{C})_\lambda$ fails, and we have to show that $(A)_\lambda$ or $(B)_\lambda$ holds.

Let

$(*)_0$ $\mu :=$ Min$\{\mu : \mathrm{cf}(\mu) \le \lambda^+, \lambda^+ < \mu \le 2^\lambda$ and $\mathrm{pp}(\mu) = \chi^*\}$.

We know by [Sh:g, II,5.4] that μ exists and (by [Sh:g, II,2.3](2)) clause (γ) holds, also $2^\lambda < \mathrm{pp}(\mu) \le \mu^{\mathrm{cf}(\mu)} \le (2^\lambda)^{\mathrm{cf}(\mu)} = 2^{\lambda + \mathrm{cf}(\mu)}$ hence $\mathrm{cf}(\mu) = \lambda^+$. So clauses $(\alpha), (\beta), (\gamma)$ hold (of course, for clause (β) use [Sh:g, Ch.II,5.4](2)), and by (γ) + [Sh:g, VIII,§1] also clause (δ) holds.

Toward trying to prove clause (ε) let

$(*)_1$ $\Upsilon :=$ Min$\{\theta : 2^\theta \ge \mu\}$,
clearly[34]

$(*)_2$ $\alpha < \Upsilon \Rightarrow 2^{|\alpha|} < \mu$ and $\Upsilon \le \lambda$ (as $2^\lambda \ge \mu$) hence $\mathrm{cf}(\Upsilon) \le \Upsilon \le \lambda < \lambda^+ = \mathrm{cf}(\mu)$ hence $2^{<\Upsilon} < \mu$.

Let

$(*)_3$ (a) u be a closed unbounded subset of Υ of order type $\mathrm{cf}(\Upsilon)$

(b) $\mathscr{T}^* = (\bigcup\limits_{\alpha \in u} {}^\alpha 2, \triangleleft)$ is a tree with $\mathrm{cf}(\Upsilon)$ levels and $\le 2^{<\Upsilon}$ nodes.

Now we shall prove clause (ε), i.e.

$(*)_4$ there is a tree with λ nodes, $\mathrm{cf}(\Upsilon)$ levels and $\ge \mu\, \Upsilon$-branches.

<u>Case A</u>: Υ has cofinality $\aleph_0$.

In the case $\Upsilon = \aleph_0$ or just $2^{<\Upsilon} \le \lambda$ clearly there is a tree as required, i.e. $\mathscr{T}^*$ is a tree having $\le 2^{<\Upsilon} \le \lambda$ nodes. So we can assume $2^{<\Upsilon} > \lambda$ and $\Upsilon > \mathrm{cf}(\Upsilon) = \aleph_0$ hence $\langle 2^\theta : \theta < \Upsilon \rangle$ is not eventually constant.

So necessarily $(\exists \theta < \Upsilon)(2^\theta \ge \lambda)$hence $2^{<\Upsilon} > \lambda^+$ (and even $2^{<\Upsilon} \ge \lambda^{+\omega}$) and for some $\theta < \Upsilon$ we have $\lambda^{++} < 2^\theta < 2^{<\Upsilon} < \mu$. Let $\chi' = \mathrm{cov}(2^{<\Upsilon}, \lambda^{++}, \lambda^{++}, \aleph_1)$, so $\chi' \ge 2^{<\Upsilon}$ and $\chi' < \mu$ by [Sh:g, II,5.4] and clause (γ) of 11.1 which have been proved (in our present possibility).

We try to apply claim 11.3 below with $\aleph_0, \lambda^+, 2^{<\Upsilon}, \chi'$ here standing for $\theta, \kappa, \mu, \chi$ there; we have to check the assumptions of 11.3 which means $2^{<\Upsilon} > \lambda^+ > \aleph_0$ and $\chi' = \mathrm{cov}(2^{<\Upsilon}, \lambda^{++}, \lambda^{++}, \aleph_1)$, both clearly hold. So the conclusion of 11.3 holds which means that $(\chi')^{\aleph_0} \ge \mathrm{cov}((2^{<\Upsilon})^{\aleph_0}, (\lambda^{++})^{\aleph_0}, \lambda^{++}, 2)$, now $(\chi')^{\aleph_0} \le \mu^{\aleph_0} \le (2^\lambda)^{\aleph_0} = 2^\lambda$ and $(2^{<\Upsilon})^{\aleph_0} = 2^\Upsilon \ge \mu$ because presently $\mathrm{cf}(\Upsilon) = \aleph_0$ and the choice

[34] Below we show that $\Upsilon > \mathrm{cf}(\Upsilon) \Rightarrow \mathrm{cf}(\Upsilon) > \aleph_0$.

of Υ and $(\lambda^{++})^{\aleph_0} \le (2^\theta)^{\aleph_0} = 2^\theta$. So by monotonicity $2^\lambda \ge \mathrm{cov}(\mu, 2^\theta, \lambda^{++}, \aleph_1)$. But $\mathrm{cov}(2^\theta, \lambda^{++}, \lambda^{++}, \aleph_1) \le \chi' := \mathrm{cov}(2^{<\Upsilon}, \lambda^{++}, \lambda^{++}, \aleph_1) < \mu \le 2^\lambda$ by clause (γ) which we have proved (and [Sh:g, ChII,5.4]) so by transitivity of cov, see 11.2(4), also $2^\lambda \ge \mathrm{cov}(\mu, \lambda^{++}, \lambda^{++}, \aleph_1)$ contradicting the present possibility.

Case B: $\mathrm{cf}(\Upsilon) > \aleph_0$.

Let $h : \mathscr{T}^* \to 2^{<\Upsilon}$ be one-to-one, see $(*)_3(b)$. Let $\mathscr{P} \subseteq [2^{<\Upsilon}]^{\le\lambda}$ be such that every $X \in [2^{<\Upsilon}]^{\le\lambda^+}$ is included in the union of countably members of $\mathscr{P}$, exists by clause (γ) of 11.1 by [Sh:g, II.5.4].

Now for every $\bar\nu = \langle \nu_\varepsilon : \varepsilon < \Upsilon\rangle \in \lim_{\mathrm{cf}(\Upsilon)}(\mathscr{T}^*)$, for some $A_{\bar\nu} \in \mathscr{P}$ we have $\Upsilon = \sup\{\varepsilon \in u : h(\nu_\varepsilon) \in A_{\bar\nu}\}$, so for every $\mu' \in (2^{<\Upsilon}, \mu)$ for some $A \in \mathscr{P}$ we have $\mu' \le |\{\bar\nu : \bar\nu \in \lim_{\mathrm{cf}(\Upsilon)}(\mathscr{T}^*)$ and $A_{\bar\nu} = A\}|$.

Now let $\mathscr{T}'$ be the closure of A to initial segments of length $\in u$, easily $\mathscr{T}'$ is as required.

So we have proved $(*)_4$ so in possibility (2) the demand $(\alpha) - (\varepsilon)$ in $(A)_\lambda$ holds.

Sub-possibility 2α: $\chi^* < 2^{\lambda^+}$.

We shall prove $(B)_\lambda$, so by the above we are left with proving clauses $(\zeta) - (\kappa)$ when $\chi^* < 2^{\lambda^+}$. By the choice of χ^*, easily the demand in clause (ζ) (in Case B of 11.1) holds; that is let $\{u_\zeta : \zeta < \chi^*\}$ be a family of subsets of ${}^{\lambda^+>}2$, a set of cardinality 2^λ, each of cardinality λ^+ such that any other such subset is included in the union of $\le \aleph_0 < \aleph_1$ of them, exist by the choice of χ^*.

Let $\mathscr{T}_\zeta = \{\nu \restriction i : \nu \in u_\zeta$ and $i \le \ell g(\nu)\}$. Now $\langle \mathscr{T}^*_\zeta : \zeta < \chi^*\rangle$ is as required.

In clause (η), "$2^\lambda < \chi^*$" holds as we are in possibility 2α.

Also as $\mathrm{pp}(\mu) = \chi^*$ and $\mathrm{cf}(\mu) = \lambda^+$ by the choice of μ necessarily (by transitivity of pcf, i.e., [Sh:g, Ch.II,2.3](2)) we have $\mathrm{cf}(\chi^*) > \lambda^+$ but $\mu > \lambda^+$. Easily $\lambda^+ < \chi \le \chi^* \wedge \mathrm{cf}(\chi) \le \lambda^+ \Rightarrow \mathrm{pp}(\chi) \le \chi^*$ hence $\mathrm{cov}(\chi^*, \lambda^{++}, \lambda^{++}, \aleph_1) = \chi^*$ by [Sh:g, Ch.II,5.4], which gives clause (λ). Now let $\mathscr{A} \subseteq [\chi^*]^{\lambda^+}$ exemplify $\mathrm{cov}(\chi^*, \lambda^{++}, \lambda^{++}, \aleph_1) = \chi^*$ and let $\mathscr{A}' = \{B : B$ is an infinite countable subset of some $A \in \mathscr{A}\}$. So $\mathscr{A}' \subseteq [\chi^*]^{\aleph_0}$ and easily $A \in [\chi^*]^{\lambda^+} \Rightarrow (\exists B \in \mathscr{A}')(B \subseteq A)$ and $|\mathscr{A}'| \le \chi^*$ as $(\lambda^+)^{\aleph_0} \le 2^\lambda < \chi^*$ certainly there is no family of $> \chi^*$ subsets of χ^* each of cardinality λ^+ with pairwise finite intersections. But by 9.4 there is $\mathscr{A}' \subseteq [\mu_{\mathrm{unif}}(\lambda^{++}, 2^{\lambda^+})]^{\lambda^+}$ of cardinality $2^{\lambda^{++}}$ such that $A \ne B \in \mathscr{A}' \Rightarrow |A \cap B| < \aleph_0$, hence we have $\chi^* < \mu_{\mathrm{unif}}(\lambda^+, 2^\lambda)$ thus completing the proof of (θ).

Now clause (ι) follows by clauses $(\eta) + (\theta) + (\kappa)$ as $\emptyset \notin \mathrm{UnfTId}_{(\chi^*)^+} \in (\lambda^+)$ which is $(\chi^*)^+$-complete ideals, see 9.4. Note also that $\emptyset \in \mathrm{WDmTId}_{\chi^*}(\lambda^+)$ by 0.5(2) and it is a $(\chi^*)^+$ complete ideal by 0.5(4). Also by clause (α) which we have proved $2^{\lambda^+} \ne \lambda^{++}$ hence $2^{\lambda^+} \ge \lambda^{+3}$ so by clause (η) (as $\chi^* < 2^{\lambda^+}$), we have $|\lim_{\lambda^+}(\mathscr{T}_\zeta)| \ge \lambda^{+3}$ for some ζ which is well known (see [J]) to imply no normal

ideal on λ^+ is λ^{++}-saturated; i.e., clause (μ). So we have proved clauses $(\alpha) - (\lambda)$ holds, i.e. that case $(B)_\lambda$ holds.

<u>Sub-possibility 2β</u>: $\chi^* = 2^{\lambda^+}$ (and $\chi^* > 2^\lambda > \lambda^+$).

We have proved that case $(A)_\lambda$ holds, as we already defined μ and χ^* and proved clauses $(\alpha), (\beta), (\gamma), (\delta), (\varepsilon)$ so we are done. $\square_{11.1}$

It may be useful to recall (actually $\lambda^{<\kappa>_{\rm tr}} = \lambda$ suffice)

11.2 Fact. 1) Assume $\lambda > \theta \geq \kappa = \text{cf}(\kappa) \geq \kappa_1$. Then $\lambda^{<\kappa>_{\rm tr}} \leq \text{cov}(\lambda, \theta^+, \kappa^+, \kappa_1)$ recalling $\mu^{<\kappa>_{\rm tr}} = \sup\{\lim_\kappa(\mathscr{T}) : \mathscr{T}$ is a tree with $\leq \mu$ nodes and κ levels, e.g. $\mathscr{T}$ a subtree of ${}^{\kappa>}\mu\}$.
2) If $\mu > \aleph_0$ is strong limit and $\lambda > \mu$ <u>then</u> for some $\kappa < \mu$ we have $\text{cov}(\lambda, \mu, \mu, \kappa)$.
3) If $\mathscr{T} \subseteq {}^{\lambda^+>}2$ is a tree, $|\mathscr{T}| \leq \lambda^+$ and $\lambda \geq \beth_\omega$ <u>then</u> for every regular $\kappa < \beth_\omega$ large enough, we can find $\langle Y_\delta : \delta < \lambda^+, \text{cf}(\delta) = \kappa\rangle, |Y_\delta| \leq \lambda$ such that:
for every $\eta \in \lim_{\lambda^+}(\mathscr{T})$ for a club of $\delta < \lambda^+$ we have $\text{cf}(\delta) = \kappa \Rightarrow \eta \restriction \delta \in Y_\delta$.
4) [Transitivity of cov] If $\mu_3 \geq \mu_2 \geq \mu_1 \geq \theta \geq \sigma = \text{cf}(\sigma)$ and $\lambda_2 = \text{cov}(\mu_3, \mu_2, \theta, \sigma)$ and $\mu < \mu_2 \Rightarrow \lambda_1 \geq \text{cov}(\mu, \mu_1, \theta, \sigma)$ <u>then</u> $\lambda_1 + \lambda_2 \geq \text{cov}(\mu_3, \mu_1, \theta, \sigma)$.

Proof. 1) E.g. proved inside 11.1.
2) By [Sh 460] or see [Sh 829].
3) Should be clear from part (2).
4) Let $\mathscr{P}_2 \subseteq [\mu_3]^{<\mu_2}$ exemplify $\lambda_2 = \text{cov}(\mu_3, \mu_2, \theta, \sigma)$ and for each $A \in P_2$ let f_A be one to one function from $|A|$ onto A and $\mathscr{P}_{1,A} \subseteq [|A|]^{<\mu_1}$ exemplify $\lambda_1 \geq \text{cov}(|A|, \mu_1, \theta, \sigma)$. Lastly, let $\mathbf{P} = \{f_A(\alpha) : \alpha \in u\} : u \in \mathbf{P}_{1,A}, A \in \mathbf{P}_2\}$ it exemplify $\lambda_1 + \lambda_2 \geq \text{cov}(\mu_3, \mu_1, \theta, \sigma)$ as required. $\square_{11.3}$

We have used in proving 11.1 also

11.3 Observation. Assume $\mu > \kappa > \theta$.
If $\chi = \text{cov}(\mu, \kappa^+, \kappa^+, \theta^+)$ <u>then</u> $\chi^\theta \geq \text{cov}(\mu^\theta, (\kappa^\theta)^+, \kappa^+, 2)$.

Proof. Let $\mathscr{P} \subseteq [\mu]^\kappa$ exemplify $\chi = \text{cov}(\mu, \kappa^+, \kappa^+, \theta^+)$.

Let $\langle \eta_\alpha : \alpha < \mu^\theta\rangle$ list ${}^\theta\mu$. Now for $\mathscr{U} \in [\mu]^\kappa$ define $\mathscr{U}^{[*]} = \{\alpha < \mu^\theta$: if $i < \theta$ then $\eta_\alpha(i) \in \mathscr{U}\}$, and let $\mathscr{P}_1 = \mathscr{P}, \mathscr{P}_2 = \{\bigcup\limits_{i<\theta} A_i : A_i \in \mathscr{P}$ for $i < \theta\}$ and $\mathscr{P}_3 = \{\mathscr{U}^{[*]} : \mathscr{U} \in \mathscr{P}_2\}$.

So (of course $\chi \geq \mu$ as $\mu > \kappa$, hence $\chi > \kappa$)

$(*)_1$ (a) $\mathscr{P}_1 \subseteq [\mu]^\kappa$ has cardinality χ

(b) $\mathscr{P}_2 \subseteq [\mu]^{\kappa^\theta}$ has cardinality $\leq \chi^\theta$

(c) $\mathscr{P}_3 \subseteq [\mu^\theta]^{\kappa^\theta}$ has cardinality $\leq \chi^\theta + \kappa^\theta = \chi^\theta$

and

$(*)_2$ if $\mathscr{U} \in [\mu^\theta]^{\leq \kappa}$ then

(a) $\mathscr{U}' := \{\eta_\alpha(i) : \alpha \in \mathscr{U} \text{ and } i < \theta\} \in [\mu]^{\leq \kappa}$

(b) there are $A_i \in \mathscr{P}_1$ for $i < \theta$ such that $\mathscr{U}' \subseteq \bigcup\limits_{i<\theta} A_i$

(c) $\bigcup\limits_{i<\theta} A_i \in \mathscr{P}_2$

(d) $\mathscr{U} \subseteq (\bigcup\limits_{i<\theta} A_i)^{[*]} \in \mathscr{P}_3 \subseteq [\mu^\theta]^{\kappa^\theta}$.

[Why? Clause (a) holds by cardinal arithmetic, clause (b) holds by the choice of $\mathscr{P} = \mathscr{P}_1$, clause (c) holds by the definition of $\mathscr{P}_2$ and clause (d) holds by the definition of $(-)^{[*]}$ and of $\mathscr{P}_3$.]

Together we are done. $\square_{11.3}$

The following is needed when we like to get in the model theory not just many models but many models no one $\leq_{\mathfrak{K}}$-embeddable into another and even just for $\dot{I}$, (see VI. E46-4d.11).

11.4 Claim. *Assume:*

(a) $\mathrm{cf}(\mu) \leq \kappa < \mu, \kappa^+ < \theta < \chi^*$ *and* $\mathrm{pp}_\kappa(\mu) = \chi^*$, *moreover* $\mathrm{pp}_\kappa(\mu) =^+ \chi^*$

(b) $\mathbf{F}$ *is a function, with domain* $[\mu]^\kappa$, *such that: for* $a \in [\mu]^\kappa, \mathbf{F}(a)$ *is a family of* $< \theta$ *members of* $[\mu]^\kappa$

(c) F *is a function with domain* $[\mu]^\kappa$ *such that*

$$a \in [\mu]^\kappa \Rightarrow a \subseteq F(a) \in \mathbf{F}(a).$$

Then we can find pairwise distinct $a_i \in [\mu]^\kappa$ *for* $i < \chi^*$ *such that* $\mathscr{I} = \{a_i : i < \chi^*\}$ *is* $(F, \mathbf{F})$*-independent which means*

$$(*)_{F,\mathbf{F},\mathscr{I}} \qquad a \neq b \ \& \ a \in \mathscr{I} \ \& \ b \in \mathscr{I} \ \& \ c \in \mathbf{F}(a) \Rightarrow \neg(F(b) \subseteq c).$$

11.5 Remark. 1) Clearly this is a relative to Hajnal's free subset theorem [Ha61].
2) Note that we can choose $F(a) = a$.
3) Also if $\mu_1 \le \mu$, $\mathrm{cf}(\mu_1) \le \kappa \le \kappa+\theta < \mu_1$ and $\mathrm{pp}_\kappa(\mu_1) \ge \mu$ then by [Sh:g, Ch.II,2.3] the Fact for μ_1 implies the one for μ.
4) Note that if $\lambda = \mathrm{cf}([\mu]^\kappa, \subseteq)$ <u>then</u> for some $\mathbf{F}, F$ as in the Fact we have

- $\circledast$ if $a_i \in [\mu]^\kappa$ for $i < \lambda^+$ are pairwise distinct then not every pair $\{a_i, a_j\}$ is $(\mathbf{F}, F)$-independent
[why? let $\mathscr{P} \subseteq [\mu]^\kappa$ be cofinal (under $\subseteq$) of cardinality λ, and let $F, \mathbf{F}$ be such that
$\mathbf{F}(a) \subseteq \{b \in [\mu]^\kappa : a \subseteq b \text{ and } b \in \mathscr{P}\}$ has a $\subseteq$-maximal member $F(a)$;
obviously there are such $F, \mathbf{F}$.

Now clearly

- $(*)_1$ if $a \ne b$ are from $[\mu]^\kappa$ and $F(a) = F(b)$ then $\{a, b\}$ is not $(\mathbf{F}, F)$-independent.

[Why? Just look at the definition of $(\mathbf{F}, F)$-independent.]

- $(*)_2$ if $\mathscr{I} \subseteq [\mu]^\kappa$ is of cardinality $> \lambda$ (e.g. λ^+) <u>then</u> $\mathscr{I}$ is not $(\mathbf{F}, F)$-independent.

[Why? As $\mathrm{Rang}(F \restriction \mathscr{I}) \subseteq \mathrm{Rang}(F) \subseteq \mathscr{P}$ and $\mathscr{P}$ has cardinality λ necessarily there are $a \ne b$ from $\mathscr{I}$ such that $F(a) = F(b)$ and use $(*)_1$.]

Proof.

- $\boxtimes_1$ it suffices to prove the variant with $[\mu]^\kappa$ replaced by $[\mu]^{\le\kappa}$.

[Why? So we are given $\mathbf{F}, F$ as in the claim. We define $g : [\mu]^{\le\kappa} \to [\mu]^\kappa$ and functions $F', \mathbf{F}'$ with domain $[\mu]^{\le\kappa}$ as follows:

$$g(a) = \{\kappa + \alpha : \alpha \in a\} \cup \{\alpha : \alpha < \kappa\}$$

$$\mathbf{F}'(a) = \{\{\alpha : \kappa + \alpha \in b\} : b \in \mathbf{F}(g(a))\}$$

$$F'(a) = \{\alpha : \kappa + \alpha \in F(g(a))\}.$$

Now $\mathbf{F}', F'$ are as in the claim only replacing everywhere $[\mu]^\kappa$ by $[\mu]^{\le\kappa}$, and if $\mathscr{I}' = \{a_i : i < \chi\} \subseteq [\mu]^{\le\kappa}$ with no repetitions satisfying $(*)_{F',\mathbf{F}',\mathscr{I}'}$ then we shall show that $\mathscr{I} := \{g(a_i) : i < \chi\}$ is with no repetitions and $(*)_{F,\mathbf{F},\mathscr{I}}$ holds.

This clearly suffices, but why it holds? Clearly g is a one-to-one function so $i \neq j < \chi \Rightarrow g(a_i) \neq g(a_j)$ and $\text{Rang}(g) \subseteq [\mu]^\kappa$ so $g(a_i) \in [\mu]^\kappa$. Let $i \neq j$ and we should check that $[c' \in \mathbf{F}(g(a_i)) \Rightarrow F(g(a_j)) \nsubseteq c']$, so fix c' such that $c' \in \mathbf{F}(g(a_i))$.

By the definition of $\mathbf{F}'(a_i)$ clearly $c := \{\alpha : \kappa + \alpha \in c'\}$ belongs to $\mathbf{F}'(a_i)$. By the choice of $\mathscr{I}' = \{a_i : i < \chi\}$ we know that $c \in \mathbf{F}'(a_i) \Rightarrow F'(a_j) \nsubseteq c$, but by the previous sentence the antecedent hold hence $F'(a_j) \nsubseteq c$ hence we can choose $\alpha \in F'(a_j) \backslash c$. By the choice of $F'(a_j)$ we have $\kappa + \alpha \in F(g(a_j))$ and by the choice of c we have $\kappa + \alpha \notin c'$, so α witness $F(g(a_j)) \nsubseteq c'$ as required.]

So we conclude that we can replace $[\mu]^\kappa$ by $[\mu]^{\leq\kappa}$. In fact we shall find the a_i in $[\mu]^{|\mathfrak{a}|}$ where $\mathfrak{a}$ chosen below.
As μ is a limit cardinal $\in (\kappa, \chi^*)$, if $\theta < \mu$ then we can replace θ by θ^+ but $\kappa^{++} < \mu$ so without loss of generality $\kappa^{++} < \theta$.

Now we prove

$\boxtimes_2$ for some unbounded subset w of χ we have $\langle \text{Rang}(f_\alpha) : \alpha \in w\rangle$ is $(\mathbf{F}, F)$-independent <u>when</u>:

$\oplus_{\chi,\mathfrak{a},\bar{f}}$ $\theta < \chi = \text{cf}(\Pi\mathfrak{a}/J)$ where $\mathfrak{a} \subseteq \mu \cap \text{Reg}\backslash\kappa^+, |\mathfrak{a}| \leq \kappa, \sup(\mathfrak{a}) = \mu, J^{\text{bd}}_{\mathfrak{a}} \subseteq J$ and for simplicity $\chi = \max \text{pcf}(\mathfrak{a})$ and $\bar{f} = \langle f_\alpha : \alpha < \chi\rangle$ is a sequence of members of $\Pi\mathfrak{a}, <_J$-increasing, and cofinal in $(\Pi\mathfrak{a}, <_J)$, so, of course, $\chi \leq \chi^*$.

Without loss of generality $f_\alpha(\lambda) > \sup(\mathfrak{a} \cap \lambda)$ for $\lambda \in \mathfrak{a}$.

Also for every $a \in [\mu]^\kappa$, define $\text{ch}_a \in \Pi\mathfrak{a}$ by $\text{ch}_a(\lambda) = \sup(a \cap \lambda)$ for $\lambda \in \mathfrak{a}$ so for some $\zeta(a) < \chi$ we have $\text{ch}_a <_J f_{\zeta(a)}$ (as $\langle f_\alpha : \alpha < \chi\rangle$ is cofinal in $(\Pi\mathfrak{a}, <_J)$). So for each $a \in [\mu]^\kappa$, as $|\mathbf{F}(a)| < \theta < \chi = \text{cf}(\chi)$ clearly $\xi(a) := \sup\{\zeta(b) : b \in \mathbf{F}(a)\}$ is $< \chi$, and clearly $(\forall b \in \mathbf{F}(a))[\text{ch}_b <_J f_{\xi(a)}]$. So $C := \{\gamma < \chi : \text{for every } \beta < \gamma, \xi(\text{Rang}(f_\beta)) < \gamma\}$ is a club of χ.

For each $\alpha < \chi$, $\text{Rang}(f_\alpha) \in [\mu]^\kappa$, hence $\mathbf{F}(\text{Rang}(f_\alpha))$ has cardinality $< \theta$, but $\theta < \chi = \text{cf}(\chi)$ hence for some $\theta_1 < \theta$ we have $\theta_1 > \kappa^+$ and $\chi = \sup\{\alpha < \chi : |\mathbf{F}(\text{Rang}(f_\alpha)| \leq \theta_1\}$, so without loss of generality $\alpha < \chi \Rightarrow \theta_1 \geq |\mathbf{F}(\text{Rang}(f_\alpha))|$.

As $\kappa^+ < \theta_1$, by [Sh 420, §1] there are $\bar{d}, S$ such that

$(*)_1$ (a) $S \subseteq \theta_1^+$ is a stationary

(b) $S \subseteq \{\delta < \theta_1^+ : \text{cf}(\delta) = \kappa^+\}$

(c) S belongs to $\check{I}[\theta_1^+]$,

(d) $\langle d_i : i < \theta_1^+\rangle$ witness it, so $\text{otp}(d_i) \leq \kappa^+, d_i \subseteq i, [j \in d_i \Rightarrow d_j = d_i \cap i]$ and $i \in S \Rightarrow i = \sup(d_i)$,
and for simplicity (see [Sh:g, III])

(e) for every club E of θ_1^+ for stationarily many $\delta \in S$ we have $(\forall \alpha \in d_\delta)[(\exists \beta \in E)(\sup(\alpha \cap d_\delta) < \beta < \alpha)]$.

Now try to choose by induction on $i < \theta_1^+$, a triple (g_i, α_i, w_i) such that:

$(*)_2$ (a) $g_i \in \Pi\mathfrak{a}$

(b) if $j < i$ then[35] $g_j <_J g_i$

(c) $(\forall \lambda \in \mathfrak{a})(\sup\limits_{j \in d_i} g_j(\lambda) < g_i(\lambda))$

(d) $\alpha_i < \chi$ and $\alpha_i > \sup(\bigcup\limits_{j<i} w_j)$

(e) $j < i \Rightarrow \alpha_j < \alpha_i$

(f) $g_i <_J f_{\alpha_i}$

(g) $\beta \in \bigcup\limits_{j<i} w_j \Rightarrow \xi(\mathrm{Rang}(f_\beta)) < \alpha_i \ \& \ f_\beta <_J g_i$

(h) w_i is a maximal subset of (α_i, χ) satisfying

$(*)$ $\beta \in w_i \ \& \ \gamma \in w_i \ \& \ \beta \neq \gamma \ \& \ a \in \mathbf{F}(\mathrm{Rang}(f_\beta)) \Rightarrow \neg(F(\mathrm{Rang}(f_\gamma)) \subseteq a)$

and moreover

$(*)^+$ $\beta \in w_i \ \& \ \gamma \in w_i \ \& \ \beta \neq \gamma \ \& \ a \in \mathbf{F}(\mathrm{Rang}(f_\beta)) \Rightarrow \{\lambda \in \mathfrak{a} : f_\gamma(\lambda) \in a\} \in J$.

Note that really (as indicated by the notation)

$\otimes$ if $w \subseteq (\alpha_i, \chi)$ satisfies $(*)^+$ then it satisfies $(*)$.

[Why? let us check $(*)$, so let $\beta \in w, \gamma \in w, \beta \neq \gamma$ and $a \in \mathbf{F}(\mathrm{Rang}(f_\beta))$; by $(*)^+$ we know that $\mathfrak{a}' = \{\lambda \in \mathfrak{a} : f_\gamma(\lambda) \in a\} \in J$. Now as J is a proper ideal on $\mathfrak{a}$ clearly for some $\lambda \in \mathfrak{a}$ we have $\lambda \notin \mathfrak{a}'$, hence $f_\gamma(\lambda) \notin a$ but $f_\gamma(\lambda) \in \mathrm{Rang}(f_\gamma)$ and by the assumption on $(\mathbf{F}, F)$ we have $\mathrm{Rang}(f_\gamma) \subseteq F(\mathrm{Rang}(f_\gamma))$ hence $f_\gamma(\lambda) \in F(\mathrm{Rang}(f_\gamma)) \backslash a$ so $\neg(F(\mathrm{Rang}(f_\gamma)) \subseteq a)$, as required.]

We claim that we cannot carry the induction because if we succeed, then as $\mathrm{cf}(\chi) = \chi > \theta \geq \theta_1^+$ there is α such that $\bigcup\limits_{i<\theta_1^+} \alpha_i < \alpha < \chi$ and let $\mathbf{F}(\mathrm{Rang}(f_\alpha)) = \{a_\zeta^\alpha : \zeta < \theta_1\}$ (possible as $1 \leq |\mathbf{F}(\mathrm{Rang}(f_\alpha))| \leq \theta_1$). Now for each $i < \theta_1^+$, by the choice of w_i clearly $w_i \cup \{\alpha\}$ does not satisfy the demand in clause (h) and

[35] in fact, without loss of generality $\min(\mathfrak{a}) > \theta_1^+$, so we can demand $g_j < g_i$ so clause (c) is redundant

let it be exemplified by some pair (β_i, γ_i). Now $\{\beta_i, \gamma_i\} \subseteq w_i$ is impossible by the choice of w_i, i.e. as w_i satisfies clause (h). Also $\beta_i \in w_i \wedge \gamma_i = \alpha$ is impossible as $\beta_i \in w_i \Rightarrow \xi(\mathrm{Rang}(f_{\beta_i})) < \alpha_{i+1} < \alpha$, so necessarily $\gamma_i \in w_i$ and $\beta_i = \alpha$, so for some $a' \in \mathbf{F}(\mathrm{Rang}(f_{\beta_i})) = \mathbf{F}(\mathrm{Rang}(f_\alpha))$ the conclusion of $(*)^+$ fails, so as $\langle a^\alpha_\zeta : \zeta < \theta_1\rangle$ list $\mathbf{F}(\mathrm{Rang}(f_\alpha))$ it follows that for some $\zeta_i < \theta_1$ we have

$$\mathfrak{a}_i = \{\lambda \in \mathfrak{a} : f_{\gamma_i}(\lambda) \in a^\alpha_{\zeta_i}\} \notin J.$$

[why use the ideal? In order to show below that $\mathfrak{b}_\varepsilon \neq \emptyset$.] But $\mathrm{cf}(\theta_1^+) = \theta_1^+ > \theta_1$, so for some $\zeta(*) < \theta_1^+$ we have $A := \{i : \zeta_i = \zeta(*)\}$ is unbounded in θ_1^+. Hence $E = \{\alpha < \theta_1^+ : \alpha$ a limit ordinal and $A \cap \alpha$ is unbounded in $\alpha\}$ is a club of θ_1^+. So for some $\delta \in S$ we have $\delta = \sup(A \cap \delta)$, moreover letting $\{\alpha_\varepsilon : \varepsilon < \kappa^+\}$ list d_δ in increasing order, we have $(\forall \varepsilon)[E \cap (\sup\limits_{\zeta<\varepsilon} \alpha_\zeta, \alpha_\varepsilon) \neq \emptyset]$ hence we can find $i(\delta,\varepsilon) \in (\sup\limits_{\zeta<\varepsilon} \alpha_\zeta, \alpha_\varepsilon) \cap A$ for each $\varepsilon < \kappa^+$.

Clearly for each $\varepsilon < \kappa^+$

$$\begin{aligned}\mathfrak{b}_\varepsilon = \big\{\lambda \in \mathfrak{a} : g_{i(\delta,\varepsilon)}(\lambda) < f_{\alpha_{i(\delta,\varepsilon)}}(\lambda) < f_{\gamma_{i(\delta,\varepsilon)}}(\lambda) \\ < g_{i(\delta,\varepsilon)+1}(\lambda) < f_{\alpha_{i(\delta,\varepsilon)+1}}(\lambda) < f_\alpha(\lambda)\big\} = \mathfrak{a} \bmod J\end{aligned}$$

hence $\mathfrak{b}_\varepsilon \cap \mathfrak{a}_{i(\delta,\varepsilon)} \notin \emptyset$. Moreover, $\mathfrak{b}_\varepsilon \cap \mathfrak{a}_{i(\delta,\varepsilon)} \notin J$. Now for each $\lambda \in \mathfrak{a}$ let $\varepsilon(\lambda)$ be $\sup\{\varepsilon < \kappa^+ : \lambda \in \mathfrak{b}_\varepsilon \cap \mathfrak{a}_{i(\delta,\varepsilon)}\}$ and let $\varepsilon(*) = \sup\{\varepsilon(\lambda) : \lambda \in \mathfrak{a}$ and $\varepsilon(\lambda) < \kappa^+\}$ so as $|\mathfrak{a}| \leq \kappa$ clearly $\varepsilon(*) < \kappa^+$. Let $\lambda^* \in \mathfrak{b}_{\varepsilon(*)+1} \cap \mathfrak{a}_{i(\delta,\varepsilon(*)+1)}$, so $B := \{\varepsilon < \kappa^+ : \lambda^* \in \mathfrak{b}_\varepsilon \cap \mathfrak{a}_{i(\delta,\varepsilon)}\}$ is unbounded in $\kappa^+, \langle f_{\beta_{i(\delta,\varepsilon)}}(\lambda^*) : \varepsilon \in B\rangle$ is strictly increasing (see clause (c) above and the choice of $\mathfrak{b}_\varepsilon$) and $\varepsilon \in B \Rightarrow f_{\beta_{i(\delta,\varepsilon)}}(\lambda^*) \in a^\alpha_{\zeta(*)}$ (by the definition of $\mathfrak{a}_{i(\delta,\varepsilon)}$, and $\zeta(*)$ as $\zeta_{i(\delta,\varepsilon)} = \zeta(*)$). We get contradiction to $a \in \mathbf{F}(\mathrm{Rang}(f_\alpha)) \Rightarrow |a| \leq \kappa$.

So really we cannot carry the induction in $(*)_2$ so we are stuck at some $i < \theta_1^+$. If $i = 0$, or i limit, or $i = j+1$ & $\sup(w_j) < \chi$ we can find g_i and then α_i and then w_i as required. So necessarily $i = j+1, \sup(w_j) = \chi$. So we have finished proving $\boxtimes_2$.

$\boxtimes_3$ there is $\mathscr{I} \subseteq [\mu]^{\leq\kappa}$ as required.

Now if χ^* is regular, recalling that we assume $\mathrm{pp}_\kappa(\mu) =^+ \chi^*$ there are $\mathfrak{a}, J$ as required in $\oplus$ above for $\chi = \chi^*$, hence also such $\bar{f}$. Applying $\boxtimes_2$ to $(\chi, \mathfrak{a}, J, \bar{f})$ we get w as there. Now $\langle \mathrm{Rang}(f_\alpha) : \alpha \in w\rangle$ is as required in the fact. So the only case left is when χ^* is singular. Let $\chi^* = \sup\limits_{\varepsilon < \mathrm{cf}(\chi^*)} \chi_\varepsilon$ and $\chi_\varepsilon \in (\mu, \chi^*) \cap$ Reg is (strictly)

increasing with ε. By [Sh:g, Ch.II,§3] we can find, for each $\varepsilon < \operatorname{cf}(\chi^*), \mathfrak{a}_\varepsilon, J_\varepsilon, \bar{f}^\varepsilon = \langle f^\varepsilon_\alpha : \alpha < \chi_\varepsilon \rangle$ satisfying the demands in $\oplus$ above, but in addition

- $\odot$ $\bar{f}^\varepsilon$ is μ^+-free i.e. for every $u \in [\chi_\varepsilon]^\mu$ there is a sequence $\langle \mathfrak{b}_\alpha : \alpha \in u \rangle$ such that $\mathfrak{b}_\alpha \in J_\varepsilon$ and for each $\lambda \in \mathfrak{a}_\varepsilon, \langle f^\varepsilon_\alpha(\lambda) : \alpha$ satisfies $\lambda \notin \mathfrak{b}_\alpha \rangle$ is strictly increasing.

So for every $a \in [\mu]^{\le \kappa}$ and $\varepsilon < \operatorname{cf}(\chi^*)$ we have

$$\big\{\alpha < \chi_\varepsilon : \{\lambda \in \mathfrak{a}_\varepsilon : f_\alpha(\lambda) \in a\} \notin J_\varepsilon\big\} \text{ has cardinality } \le \kappa.$$

Hence for each $a \in [\mu]^{\le \kappa}$

$$\big\{(\varepsilon, \alpha) : \varepsilon < \operatorname{cf}(\chi^*) \text{ and } \alpha < \chi_\varepsilon \text{ and } \{\lambda \in \mathfrak{a}_\varepsilon : f_\alpha(\lambda) \in a\} \notin J_\varepsilon\big\}$$

has cardinality $\le \kappa + \operatorname{cf}(\chi^*) = \operatorname{cf}(\chi^*)$ as for singular $\mu > \kappa \ge \operatorname{cf}(\mu)$ we have $\operatorname{cf}(\mathrm{pp}_\kappa(\mu)) > \kappa$.

Define: $X = \{(\varepsilon, \alpha) : \varepsilon < \operatorname{cf}(\chi^*), \alpha < \chi_\varepsilon\}$

$$F'\big((\varepsilon,\alpha)\big) = \big\{(\varepsilon',\alpha') : (\varepsilon',\alpha') \in X \backslash \{(\varepsilon,\alpha)\} \text{ and for some } \\ d \in \mathbf{F}(\mathrm{Rang}(f^\varepsilon_\alpha)) \text{ we have } \{\lambda \in \mathfrak{a}_\varepsilon : f^{\varepsilon'}_{\alpha'}(\lambda) \in d\} \notin J_{\varepsilon'}\big\}$$

so $F'\big((\varepsilon,\alpha)\big)$ is a subset of X of cardinality $< \operatorname{cf}(\chi^*)^+ + \theta < \chi^*$.

So by Hajnal's free subset theorem [Ha61] we finish proving $\boxtimes_3$ (we could alternatively, for χ^* singular, have imitated his proof).

Recalling $\boxtimes_1$ we are done. $\square_{11.4}$

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